VIBRATION PROBLEMS IN ENGINEERING

TIMOSHENKO

$5

~~Joseph C Pafferle~~
~~2735 Haste St.~~
~~Berk.~~

Peter I. Brown
701 - Paloma Ave
Oakland, California
Glencourt 0890.

WILLIAM FELZER
2925 Rivera St.
San Francisco 16, Calif.

WILLIAM FELZER
2925 Rivera St.
San Francisco 16, Calif.

VIBRATION PROBLEMS IN ENGINEERING

By

S. TIMOSHENKO

Professor of Engineering Mechanics
University of Michigan

SECOND PRINTING

NEW YORK
D. VAN NOSTRAND COMPANY, Inc.
250 Fourth Avenue

Copyright, 1928, by
D. VAN NOSTRAND COMPANY, Inc.

All rights reserved, including that of translation into the Scandinavian and other foreign languages.

First Published . . October, 1928
Second Printing . September, 1929

PRINTED IN THE UNITED STATES OF AMERICA

PREFACE

With the increase of size and velocity in modern machines, the analysis of vibration problems becomes more and more important in mechanical engineering design. It is well known that problems of great practical significance, such as the balancing of machines, the torsional vibration of shafts and of geared systems, the vibrations of turbine blades and turbine discs, the whirling of rotating shafts, the vibrations of railway track and bridges under the action of rolling loads, the vibration of foundations, can be thoroughly understood only on the basis of the theory of vibration. Only by using this theory can the most favorable design proportions be found which will remove the working conditions of the machine as far as possible from the critical conditions at which heavy vibrations may occur.

In the present book, the fundamentals of the theory of vibration are developed, and their application to the solution of technical problems is illustrated by various examples, taken, in many cases, from actual experience with vibration of machines and structures in service. In developing this book, the author has followed the lectures on vibration given by him to the mechanical engineers of the Westinghouse Electric and Manufacturing Company during the year 1925, and also certain chapters of his previously published book on the theory of elasticity.*

The contents of the book in general are as follows:

The first chapter is devoted to the discussion of harmonic vibrations of systems with one degree of freedom. The general theory of free and forced vibration is discussed, and the application of this theory to balancing machines and vibration-recording instruments is shown. The Rayleigh approximate method of investigating vibrations of more complicated systems is also discussed, and is applied to the calculation of the whirling speeds of rotating shafts of variable cross-section.

Chapter two contains the theory of the non-harmonic vibration of systems with one degree of freedom. The approximate methods for investigating the free and forced vibrations of such systems are discussed. A particular case in which the flexibility of the system varies with the time is considered in detail, and the results of this theory are applied to the investigation of vibrations in electric locomotives with side-rod drive.

* *Theory of Elasticity*, Vol. II (1916)—St. Petersburg, Russia.

In chapter three, systems with several degrees of freedom are considered. The general theory of vibration of such systems is developed, and also its application in the solution of such engineering problems as: the vibration of vehicles, the torsional vibration of shafts, whirling speeds of shafts on several supports, and vibration absorbers.

Chapter four contains the theory of vibration of elastic bodies. The problems considered are: the longitudinal, torsional, and lateral vibrations of prismatical bars; the vibration of bars of variable cross-section; the vibrations of bridges, turbine blades, and ship hulls; the theory of vibration of circular rings, membranes, plates, and turbine discs.

Brief descriptions of the most important vibration-recording instruments which are of use in the experimental investigation of vibration are given in the appendix.

The author owes a very large debt of gratitude to the management of the Westinghouse Electric and Manufacturing Company, which company made it possible for him to spend a considerable amount of time in the preparation of the manuscript and to use as examples various actual cases of vibration in machines which were investigated by the company's engineers. He takes this opportunity to thank, also, the numerous friends who have assisted him in various ways in the preparation of the manuscript, particularly Messr. J. M. Lessells, J. Ormondroyd, and J. P. Den Hartog, who have read over the complete manuscript and have made many valuable suggestions.

He is indebted, also, to Mr. F. C. Wilharm for the preparation of drawings, and to the Van Nostrand Company for their care in the publication of the book.

S. Timoshenko

Ann Arbor, Michigan,
May 22, 1928.

CONTENTS

CHAPTER I

HARMONIC VIBRATIONS OF SYSTEMS HAVING ONE DEGREE OF FREEDOM

1. Free Harmonic Vibrations 1
2. Illustrations 4
3. Forced Vibrations 8
4. Technical Application (Simple Harmonic Motion) 13
5. Another Method of Calculating Forced Vibrations 19
6. Damping Proportional to the Velocity 21
7. Constant Damping 25
8. Forced Vibrations with Damping 26
9. Technical Applications (Forced Vibrations with Damping) 33
10. Balancing of Rotating Machines 38
11. Machines for Balancing 40
12. General Case of Disturbing Force 47
13. The Energy Method of Studying Vibration Problems 52
14. Rayleigh Method 55
15. Critical Speed of a Rotating Shaft 61

CHAPTER II

NON-HARMONIC VIBRATIONS OF SYSTEMS HAVING ONE DEGREE OF FREEDOM

16. General Considerations 67
17. Examples of Pseudoharmonic Vibration 68
18. Free Pseudoharmonic Vibrations 72
19. Graphical Solution 74
20. Numerical Solution 79
21. Forced Pseudoharmonic Vibrations 81
22. Quasiharmonic Vibrations 87
23. Vibrations in the Side-rod Drive System of Electric Locomotives 91

CHAPTER III

SYSTEMS HAVING SEVERAL DEGREES OF FREEDOM

24. d'Alembert's Principle and the Principle of Virtual Displacements 109
25. Generalized Coordinates and Generalized Forces 112
26. Lagrange's Equations 116
27. The Spherical Pendulum 119
28. Free Vibrations. General Discussion 121
29. Forced Vibrations 126
30. Vibration of Vehicles 128
31. Torsional Vibrations of Shafts and Geared Systems 138
32. Lateral Vibrations of Shafts on Many Supports 159
33. Gyroscopic Effects on the Critical Speeds of Rotating Shafts 172
34. Effect of Weight of Shaft and Discs on the Critical Speed 184
35. Effect of Flexibility of Shafts on the Balancing of Machines 189
36. The Dynamic Vibration Absorber 191

CHAPTER IV

Vibration of Elastic Bodies
37. Longitudinal Vibrations of Prismatical Bars 199
38. Vibration of a Bar with a Load at the End 209
39. Torsional Vibration of Circular Shafts 216
40. Lateral Vibration of Prismatical Bars 221
41. The Effect of Shearing Force and Rotatory Inertia 226
42. Free Vibration of a Bar with Hinged Ends 228
43. Other End Fastenings 232
44. Forced Vibration of a Beam with Supported Ends 237
45. Vibration of Bridges 247
46. Effect of Axial Forces on Lateral Vibrations 253
47. Vibration of Beams on Elastic Foundation 256
48. Ritz Method 259
49. Vibration of Rods of Variable Cross Section 264
50. Vibration of Turbine Blades 270
51. Vibration of Hulls of Ships 276
52. Lateral Impact of Bars 279
53. Longitudinal Impact of Prismatical Bars 284
54. Vibration of a Circular Ring 292
55. Vibration of Membranes 297
56. Vibration of Plates 307
57. Vibration of Turbine Discs 320

APPENDIX

Vibration Measuring Instruments
1. General 327
2. Frequency Measuring Instruments 327
3. The Measurement of Amplitudes 328
4. Seismic Vibrographs 332
5. Torsiograph 336
6. Torsion Meters 337
7. Strain Recorders 341
Index 347

CHAPTER I

HARMONIC VIBRATIONS OF SYSTEMS HAVING ONE DEGREE OF FREEDOM

1. Free Harmonic Vibrations. If an elastic system, such as a loaded beam, a twisted shaft or a deformed spring, be disturbed from its position of equilibrium by an impact or by the sudden application and removal of an additional force, the elastic forces of the member in the disturbed position will no longer be in equilibrium with the loading, and vibrations will ensue. Generally an elastic system can perform vibrations of different modes. For instance, a string or a beam while vibrating may assume the different shapes depending on the number of nodes subdividing the length of the member. In the simplest cases the configuration of the vibrating system can be determined by one quantity only. Such systems are called *systems having one degree of freedom.*

Let us consider the simplest case shown in Fig. 1. If the arrangement be such that only vertical displacements of the weight W are possible and the mass of the spring be small in comparison with that of the weight W, the system can be considered as having *one degree of freedom.* The configuration will be determined completely by the vertical displacement of the weight.

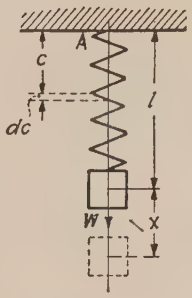


Fig. 1.

By an impulse or a sudden application and removal of an external force vibrations of the system can be produced. Such vibrations which are maintained by the elastic force in the spring alone are called *free* or *natural* vibrations. An analytical expression for these vibrations can be found from the differential equation of motion, which always can be written down if the forces acting on the moving body are known.

Let k denote the load necessary to produce a unit extension of the spring. Then the static deflection of the spring under the action of the weight W will be

$$\delta_{st} = \frac{W}{k}.$$

Denoting a vertical displacement of the weight from its position of equilibrium by x and considering this displacement as positive if it is in

a downward direction, the expression for the tensile force in the spring corresponding to any position of the weight becomes

$$F = W + kx. \tag{a}$$

In order to derive the differential equation of motion d'Alembert's principle will be used. On the basis of this principle the equation of motion can be obtained in the same manner as an equation of statics. It is only necessary in addition to the external and elastic forces to take into consideration the forces of inertia. In the present case the inertia force is given by the expression $-\frac{W\ddot{x}}{g}$ in which $\ddot{x}$, the second derivative of the displacement x with respect to time t, represents the acceleration of the moving load W and g = acceleration due to gravity. Adding the inertia force to the load W and equating the sum to the force F in the spring we have,

$$W - \frac{W\ddot{x}}{g} = W + kx, \tag{1}$$

from which

$$\ddot{x} + p^2x = 0, \tag{2}$$

where

$$p^2 = \frac{kg}{W} = \frac{g}{\delta_{st}}. \tag{3}$$

Equation (2) will be satisfied if we put $x = A \cos pt$ or $x = B \sin pt$, where A and B are arbitrary constants. By adding these solutions the general solution of equation (2) will be obtained:

$$x = A \cos pt + B \sin pt. \tag{4}$$

It is seen that the vertical motion of the weight W has a vibratory character. The *period* of this vibration is

$$\tau = \frac{2\pi}{p}$$

and the *frequency*

$$f = \frac{1}{\tau} = \frac{p}{2\pi}.$$

Substituting for p its expression (3) we have

$$\tau = 2\pi\sqrt{\frac{\delta_{st}}{g}}; \qquad f = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_{st}}}, \tag{5}$$

i.e., the period of vibration of the load on the spring is the same as that of a mathematical pendulum, the length of which is equal to δ_{st}. The *amplitude* of vibrations (4), i.e., the maximum displacement of the load

from the position of equilibrium, remains constant because A and B are constant. A vibratory motion represented by equation (4) is called a *harmonic motion.*

In order to determine the arbitrary constants A and B, the initial conditions must be considered. Assume, for instance, that at the initial moment ($t = 0$) the weight W has a displacement x_0 from its position of equilibrium and that its initial velocity is $\dot{x}_0$. Substituting $t = 0$ in equation (4) we obtain

$$x_0 = A. \tag{a}$$

Taking now the derivative of eq. (4) with respect to time and substituting in this derivative $t = 0$, we have

$$\frac{\dot{x}_0}{p} = B. \tag{b}$$

Substituting the values of the arbitrary constants (a) and (b), the following expression for the vibratory motion of the weight W will be obtained:

$$x = x_0 \cos pt + \frac{\dot{x}_0}{p} \sin pt. \tag{6}$$

It is seen that for the general case the vibration consists of two parts; a vibration which is proportional to *cos pt* and depends on the initial displacement of the system and another which is proportional to *sin pt* and depends on the initial speed $\dot{x}_0$. Such a motion as (6) can always be represented as a simple sinusoidal motion. Let,

$$x_0 = M \sin \alpha, \tag{c}$$

$$\frac{\dot{x}_0}{p} = M \cos \alpha, \tag{d}$$

where M and α are constants to be determined.

Substituting in eq. (6) we have

$$x = M \sin (pt + \alpha). \tag{7}$$

The *amplitude* M of this vibration and the angle α, the *phase* of the vibration, can be easily found from equations (c) and (d). Squaring and adding these equations we obtain,

$$M^2 = x_0^2 + \frac{\dot{x}_0^2}{p^2}$$

or

$$M = \sqrt{x_0^2 + \frac{\dot{x}_0^2}{p^2}}\,. \tag{8}$$

Dividing (*c*) by (*d*), we obtain

$$\tan \alpha = \frac{x_0 p}{\dot{x}_0}\,. \tag{9}$$

The amplitude M and the angle α can also be obtained graphically as shown in Fig. 2. Taking $\overline{OA} = \dot{x}_0/p$ and $\overline{OB} = x_0$ we have, from the right-angled triangle OAC,

$$\overline{OC} = \sqrt{x_0^2 + \frac{\dot{x}_0^2}{p^2}} = M$$

and

$$\tan \alpha = \frac{\overline{AC}}{\overline{AO}} = \frac{x_0 p}{\dot{x}_0}\,.$$

Imagine now that the radius $\overline{OC}$ is moving with a constant angular velocity p in clockwise direction, then after a time t the point C will be at C_1 and the vertical projection $\overline{OB_1}$ of the radius $\overline{OC_1}$, equal to $M \sin (pt + \alpha)$, represents the vertical displacement x (see eq. 7) of the load W during the vibration.

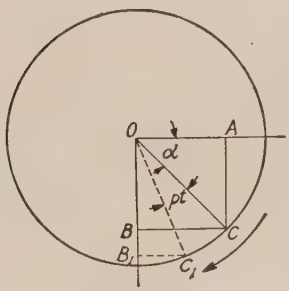


Fig. 2.

It is therefore seen that the natural vibration of the system shown in Fig. 1 can always be represented as a simple harmonic vibration, the *period* of which (see eq. 5) depends on δ_{st}, i.e., on the rigidity of the spring and the weight of the load. The amplitude and phase depend on the initial displacement and initial velocity of the load.

2. Illustrations.—The following examples will serve to illustrate the various points of the preceding theory and to show how the period of natural vibration of a system can be calculated for a variety of cases.

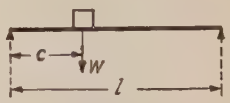


Fig. 3.

a. Vibration of a Load W on a Beam (Fig. 3).—Neglecting the weight of the beam and applying the simple theory of bending the static deflection of the beam directly under the load will be

$$\delta_{st} = \frac{Wc^2(l - c)^2}{3lEI}\,. \tag{a}$$

This deflection is proportional to the load and the beam can therefore be considered as a spring, the *rigidity* of which is determined by the quantity

$$k = \frac{3lEI}{c^2(l - c)^2},$$

which represents the load necessary to produce a deflection of the beam equal to unity. Substituting δ_{st} obtained above (see eq. (a)) in eq. (5) the period of natural vibration of the load on the beam can easily be calculated. The value of the period obtained in this manner will be somewhat smaller than the true period due to the fact that the weight of the beam itself was neglected. A more detailed discussion of this subject will be given in paragraph 14.

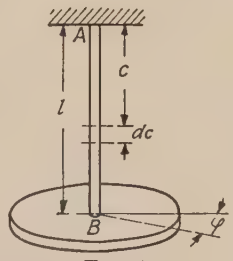


FIG. 4.

b. Torsional Vibration.—If a disc be supported as in Fig. 4 and a couple in its plane be applied and suddenly removed, free torsional vibration of the shaft with the disc will be produced. Let

φ = angle of twist of the shaft at any moment during the vibration,

k = torque moment necessary to produce an angle of twist of one radian in the shaft.

In the case of a shaft of uniform circular cross-section having a diameter d we have

$$k = \frac{G}{l}\frac{\pi d^4}{32}.$$

The torque moment in the shaft at any moment during vibration will be $k\varphi$. By neglecting the mass of the shaft and equating the torque moment in the shaft to the moment of the inertia forces of the disc the following equation of motion of the disc will be obtained

$$I\ddot{\varphi} + k\varphi = 0, \tag{10}$$

in which,

I = moment of inertia of the disc about the axis AB of the shaft perpendicular to the plane of the disc. In the case of a circular disc of constant thickness

$$I = \frac{\pi D^4 h\gamma}{32g} = \frac{WD^2}{8g},$$

in which,

h = thickness of the disc,
D = diameter of the disc,
γ = weight per unit volume of the disc,
W = weight of the disc.

If instead of a circular disc a wheel is taken and if its mass be considered as concentrated in the rim we have,

$$I = \frac{WD^2}{4g}.$$

In general for a disc of variable thickness h

$$I = \frac{2\pi}{g}\int_0^{1/2D} h\gamma r^3 dr.$$

Dividing equation (10) by I and letting

$$p^2 = \frac{k}{I}, \tag{11}$$

an equation of the same form as eq. (2) is obtained, the general solution of which will be:

$$\varphi = A \cos pt + B \sin pt.$$

It is seen that the period of torsional vibrations is

$$\tau = \frac{2\pi}{p} = 2\pi\sqrt{\frac{I}{k}}. \tag{12}$$

In the case of a circular shaft of uniform cross section:

$$\tau = 2\pi\sqrt{\frac{32Il}{G\pi d^4}}, \qquad f = \frac{1}{\tau} = \frac{1}{2\pi}\sqrt{\frac{\pi G d^4}{32Il}}. \tag{13}$$

It was assumed in this case that the shaft had a constant diameter d. When the shaft is of different diameter in different parts it can be easily reduced to one having a constant diameter. It is only necessary to bear in mind that the angle of twist of a shaft is proportional to the length and inversely proportional to the fourth degree of the diameter. Therefore, without changing the angle of twist, a part of the shaft of length a_1 and diameter d_1 may be replaced by an equivalent part of length a_2 and diameter d_2 by satisfying the condition:

$$\frac{a_1}{a_2} = \left(\frac{d_1}{d_2}\right)^4. \tag{14}$$

The results obtained can be applied also in the case of a shaft with two rotating masses (Fig. 5). Such an arrangement is of great practical importance in the field of engineering, and is encountered, for instance, in the case of a propeller shaft.* If the mass of the shaft can be neglected in comparison with the masses at the ends, the system shown in Fig. 5 can be considered as having two degrees of freedom: (1) one arising from the fact that the shaft with the discs can rotate in its bearings as a rigid body and (2) from the fact that the shaft may perform torsional vibration during which the discs are oscillating in directions opposite to each other. Imagine for instance that at the initial moment ($t = 0$) two equal and opposite twisting couples are acting at the ends of the shaft and then suddenly removed. In such a manner torsional vibrations alone will be produced and the discs will rotate in opposite directions with respect to each other so that the moment of momentum of the system with respect to the axis of the shaft will always remain equal to zero as it was in the initial moment ($t = 0$). During such vibrations a certain intermediate section mn of the shaft will remain immovable. This section is called the *nodal section*, and its position will be found from the condition that the parts of the shaft to the right and left (Fig. 5) have the same period of vibration. Applying eq. (13) to each of the two sides of the shaft we obtain

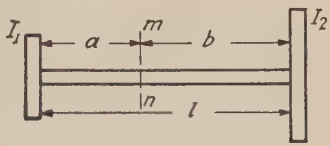


Fig. 5.

$$\sqrt{I_1 a} = \sqrt{I_2 b}$$

or

$$\frac{a}{b} = \frac{I_2}{I_1}. \qquad (a)$$

From Fig. (5) we have

$$a + b = l. \qquad (b)$$

Then, from (a) and (b),

$$a = \frac{I_2 l}{I_1 + I_2}, \qquad b = \frac{I_1 l}{I_1 + I_2}.$$

Now, from eq. (13) the period and the frequency of torsional vibrations

* This problem was studied first by H. Frahm; see "Neue Untersuchungen über dynamischen Vorgänge in den Wellenleitungen v. Schiffsmaschinen," V. D. I., 1902, p. 797. He showed experimentally that at a certain speed vibrations produce very great effects on the stresses in propeller shafts.

of the system will be

$$\tau = 2\pi\sqrt{\frac{32I_1I_2l}{G\pi d^4(I_1 + I_2)}}; \qquad f = \frac{1}{2\pi}\sqrt{\frac{\pi G d^4(I_1 + I_2)}{32I_1I_2l}}. \tag{15}$$

The effect of the mass of the shaft on the period of vibration will be considered later (see paragraph 14).

3. Forced Vibrations.—In considering the vibrations of the system shown in Fig. 1, *free vibrations* only were discussed. Another type will now be considered. If it be assumed that in addition to the constant force W, a periodically variable vertical *disturbing force* Q is acting on the load, *forced vibrations* of the system will be produced. The corresponding differential equation of motion is obtained by adding to the left side of eq. (1) the force Q. In this manner we have,

$$\frac{W}{g}\ddot{x} + kx = Q \tag{16}$$

or, dividing by W/g, and using, as before, the notation

$$p^2 = \frac{kg}{W},$$

the following equation is obtained:

$$\ddot{x} + p^2x = \frac{Qg}{W}. \tag{17}$$

The particular case that the disturbing force Q is proportional to $\cos mt$ will now be considered. The period of this force is taken $\tau_1 = 2\pi/m$ and the frequency $f_1 = m/2\pi$. Letting

$$\frac{Qg}{W} = q \cos mt,$$

eq. (17) for this case becomes

$$\ddot{x} + p^2x = q \cos mt. \tag{18}$$

If the variation of the disturbing force is very slow, i.e., m is small in comparison with p, the term containing acceleration in eq. (18) can be neglected and we obtain a statical deflection

$$x_{st} = \frac{q \cos mt}{p^2}. \tag{a}$$

In order to get the dynamical displacement, the general solution of eq. (18) must be used. This solution will be obtained if to the solution (4) of the homogeneous eq. (2) a particular solution of eq. (18) be added. Taking $C \cos mt$ as such a particular solution and substituting this in eq. (18) we obtain

$$C = \frac{q}{p^2 - m^2}$$

and the general solution of eq. (18) becomes

$$x = A \cos pt + B \sin pt + \frac{q}{p^2 - m^2} \cos mt. \tag{19}$$

In this equation the first two terms on the right side represent free vibrations of the system which were previously discussed while the last term represents *forced vibrations.* These latter have the same period as the disturbing force and their amplitude C depends on the frequency of this force. The ratio of this amplitude to the statical deflection, given by eq. (a) is called the *magnification factor* and is equal to

$$\beta = \frac{1}{1 - \dfrac{m^2}{p^2}} \qquad \text{or} \qquad \beta = \frac{1}{1 - \dfrac{\tau^2}{\tau_1^2}}, \tag{20}$$

where

$$\tau = \frac{2\pi}{p} = \text{the period of natural vibration}$$

and

$$\tau_1 = \frac{2\pi}{m} = \text{the period of the disturbing force.}$$

It is seen that for small values of m/p, i.e., when the frequency of the disturbing force is small as compared with that of the natural vibrations of the system the magnification factor β will be near to unity and the amplitude of the forced vibrations is only slightly different from the statical deflection given by eq. (a). With an increase in the ratio m/p the amplitude of the forced vibrations increases and becomes infinitely large when $m/p = 1$, i.e., when the frequency of the disturbing force becomes equal to the frequency of the natural vibrations of the system.* When this occurs we have the *condition of resonance* and the corresponding frequency of the disturbing force is called the *critical frequency.*

* For a more detailed discussion of this subject, see paragraph 29.

In engineering problems the disturbing force results very often from a condition of *unbalance.* Assume, for instance, that a machine supported by a beam AB (see Fig. 6) and having an unbalance M is rotating with a constant angular velocity m. Let Q denote the centrifugal force of the unbalance M, then $Q \cos mt$ represents the vertical disturbing force which will produce vibrations of the beam. The differential equation of these vibrations will be the same as eq. (16)* and it can be concluded at once that the condition of resonance will take place when the number of r.p.m. of the machine becomes equal to the number of natural vibrations of the beam loaded by the weight W of the machine, i.e., when

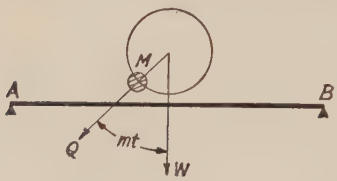


Fig. 6.

$$m = \sqrt{\frac{g}{\delta_{st}}}.$$

The speed corresponding to the condition of resonance is called the *critical speed.* If the machine is running at this speed, large vibrations of the beam will be produced whose amplitude will depend on the magnitude of damping caused by different kinds of resistances, such as resistance of the air, friction at the supports of the beam, internal friction in the

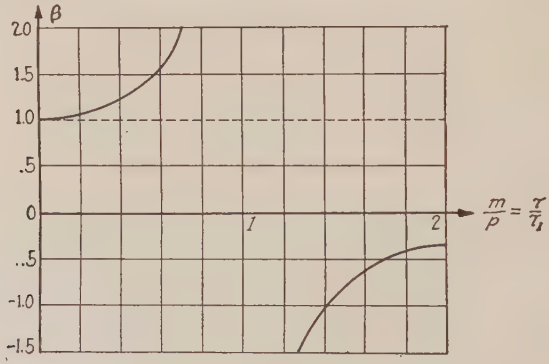


Fig. 7.

material of the beam. The effect of such damping will be discussed later. In the above analysis the damping has not been taken into con-

* The mass of the beam and the effect of the horizontal component of the centrifugal force are neglected in this consideration.

sideration and as a result of this an unlimited increase of the amplitude of the forced vibrations is obtained as m/p approaches unity.

Returning to the expression (20) it is seen that with an increase in the frequency m of the disturbing force beyond that corresponding to the natural vibrations of the system, the amplitude of the forced vibration decreases and if m becomes very large in comparison with p the amplitude of the forced vibrations becomes very small in comparison with the static deflection and the load W (Fig. 1) can be considered in this case as immovable in space. In Fig. 7 the variation of the ratio β of the amplitude of forced vibrations to the statical deflection is represented graphically as a function of $m/p = \tau/\tau_1$.

Considering the sign of the expression (20) it is seen that for the case $m < p$ this expression is positive and for $m > p$ it becomes negative. This indicates that when the frequency of the disturbing force is less than that of the natural vibration of the system the forced vibrations and the disturbing force are always in the same *phase*, i.e., the vibrating load (Fig. 1) reaches its lowest position at the same moment as the disturbing force assumes its maximum value in a downward direction. When $m > p$ the difference in phase between the forced vibration and the disturbing force becomes equal to π. This means that at the moment when the force is a maximum in a downward direction the vibrating load reaches its upper position. This phenomenon can be illustrated by the following simple experiment. In the case of a simple pendulum AB (Fig. 8) forced vibrations can be produced by giving an oscillating motion in the horizontal direction to the point A. If this oscillating motion has a frequency lower than that of the pendulum the extreme positions of the pendulum during such vibrations will be as shown in Fig. 8-*a*, the motions of the points A and B will be in the same phase. If the oscillatory motion of the point A has a higher frequency than that of the pendulum the extreme positions of the pendulum during vibration will be as shown in Fig. 8-*b*. The phase difference of the motions of the points A and B in this case is equal to π.

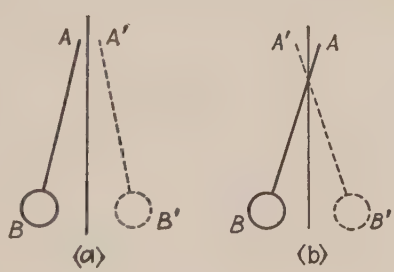


Fig. 8.

In our discussion the third term only of the general solution (19) representing forced vibrations has been considered. In applying a disturbing force, however, not only forced vibrations are produced but also

natural vibrations. After a time the latter vibrations will be damped out due to different kinds of resistance but at the beginning of motion they may have a practical importance. The amplitude of the natural vibrations can easily be found from the general solution (19) by taking the initial conditions into consideration. Assume, for instance, that at the initial moment ($t = 0$) the displacement and the velocity are equal to zero, the arbitrary constants of the solution (19) must then be determined in such a manner that

$$x = 0 \qquad \text{and} \qquad \dot{x} = 0 \qquad \text{for} \qquad t = 0.$$

These conditions will be satisfied by putting

$$A + \frac{q}{p^2 - m^2} = 0; \qquad B = 0.$$

Substituting in (19), we have

$$x = \frac{q}{p^2 - m^2}(\cos mt - \cos pt). \tag{19^1}$$

At the beginning of the action the disturbing force $q \cos mt$ produces forced vibrations and free vibrations of the same amplitude. If the frequency of the disturbing force approaches the frequency of the natural vibrations the phenomenon of *beating* takes place. Let

$$p - m = 2\Delta,$$

then, from eq. (19^1) we have

$$x = -\frac{2q}{p^2 - m^2}\sin\frac{(m+p)t}{2}\sin\frac{(m-p)t}{2} = \frac{2q\sin\Delta t}{p^2 - m^2}\sin\frac{(m+p)t}{2}. \tag{21}$$

The result can be represented graphically as shown in Fig. 9.

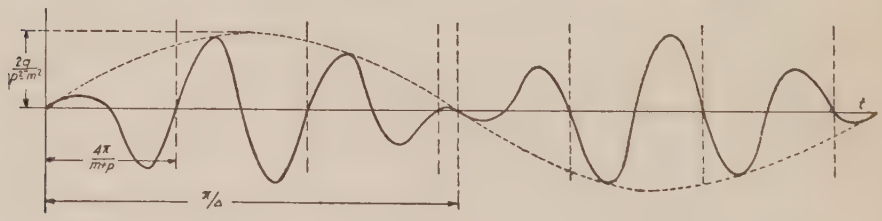


Fig. 9.

This gives vibrations having the period

$$\tau = 2\pi : \frac{m+p}{2} = \frac{4\pi}{m+p}$$

and of a variable amplitude equal to

$$\frac{2q \sin \Delta t}{p^2 - m^2}.$$

The period of *beating* will be $2\pi/\Delta$ which will increase as m approaches p and becomes infinite for the case of resonance ($p = m$). Equation (21) for this case can be represented as follows:

$$x = \frac{2q\Delta t}{p^2 - m^2} \sin \frac{m + p}{2} t = \frac{qt}{2m} \sin mt, \tag{21^1}$$

i.e., the amplitude increases indefinitely with the time. Such a condition is obtained as a result of neglecting the damping effect of various kinds of resistances.

4. Technical Applications (Simple Harmonic Motion).—In order to illustrate the previously discussed theory some simple cases will now be considered. These deal chiefly with the instruments for measuring vibrations.

a. Instruments for Recording Vibrations (Vibrographs).—The theory of the Vibrograph can easily be obtained from a consideration of the system shown in Fig. 1. Forced vibrations of the weight W can be produced by a vibratory motion of the point of suspension A of the spring in a vertical direction. Let x_1 be the displacement of the point A at any moment in a downward direction and x, the displacement of the load W at the same moment. The extension of the spring will be $x - x_1$ and the tensile force in the spring will be equal to $k(x - x_1)$. Substituting this force for kx in eq. (1), the following equation of motion is obtained;

$$\frac{W}{g} \ddot{x} + k(x - x_1) = 0. \tag{22}$$

Let point A perform a simple harmonic oscillation given by the equation

$$x_1 = b \cos mt.$$

Then letting

$$\frac{kg}{W} = p^2 \qquad \text{and} \qquad \frac{kbg}{W} = q \tag{a}$$

we obtain

$$\ddot{x} + p^2 x = q \cos mt.$$

This equation is the same as eq. (18) and all the conclusions of paragraph 3 can be applied. If the frequency of the oscillatory motion of the point of suspension A is very low as compared with the frequency of the natural

vibrations of the load W attached to the spring, the conditions of motion will only differ slightly from those of static conditions, i.e., the load W will perform approximately the same oscillatory motion as the point of suspension A. When the frequency of the oscillations of the point A approaches that of the natural vibrations of the system, heavy forced vibrations will be produced. In the case when m is very large in comparison with p, i.e., when the frequency of oscillation of the point A is very high in comparison with that of the natural vibrations, the amplitude of the forced vibrations becomes very small and the load W can be considered as immovable in space. The amplitude of the forced vibrations is, from eq. 19,

$$C = \frac{q}{p^2 - m^2}$$

or by using the equation (a) above

$$C = \frac{b}{1 - \frac{m^2}{p^2}}. \tag{23}$$

Taking, for instance, $m = 10p$, we have for the amplitude of the forced vibrations $1/99b$. This means that for such a case vibrations of the point of suspension will scarcely be transmitted to the load W. On this principle is based the use of flexible suspensions for protecting sensitive measuring instruments from high frequency vibrations transmitted to the buildings through the foundation. The same principle is also used in seismographs: instruments for recording earth vibrations during earthquakes. On this principle also are built various instruments (vibrographs) for recording vibrations of machines, buildings, foundations, bridges, and hulls of ships.

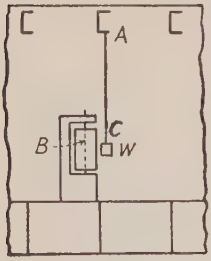


Fig. 10.

In Fig. 10 a simple arrangement is shown which can be used for recording vibrations in ship hulls. A weight W is attached to the point A of a beam by a rubber band AC. During vertical vibrations of the hull this weight remains practically immovable provided the period of natural vibrations of this weight is sufficiently large, and the pencil attached to it will record the vibrations of the hull on a rotating drum B. For obtaining the amplitude of vibration with sufficient accuracy it is necessary to have the period of the natural vibrations of the weight large in comparison with that of the hull of the ship. To satisfy this condi-

tion the length of the string AC and its elongation must be comparatively large. For instance, to obtain a natural period of vibration of 2 seconds the elongation of the string under the action of the weight W must be nearly 3 ft. (see eq. (5)). This large extension is a defect in this type of arrangement.

An instrument for recording ship vibrations in which a much smaller elongation of the spring is necessary will now be considered. This arrangement is shown in Fig. 11.* Neglecting the weight of the lever

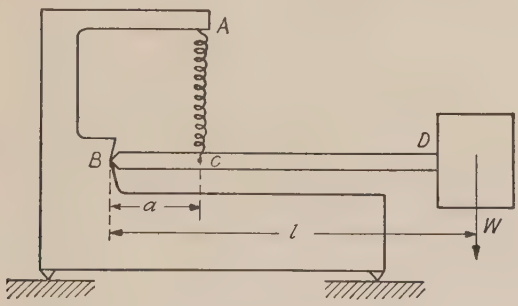


Fig. 11.

BD in comparison with the weight W, the elongation of the spring AC is Wl/ka and the statical deflection of the weight will be

$$\delta_{st} = \frac{Wl^2}{ka^2}.$$

This deflection must be substituted in eq. (5) for obtaining the period of natural vibration of the vibrograph. We see that for the same period the necessary extension of the spring in this case will be diminished in the ratio a/l compared to that of the arrangement shown in Fig. 10.†

b. Frahm's Vibration Tachometer.‡—An instrument widely used for measuring the frequency of vibrations is known as Frahm's tachometer. This consists of a system of steel strips built in at their lower ends as shown in Fig. 12 (*b*). To the upper ends of the strips small masses are attached, the magnitudes of which are adjusted in such a manner that

* Such an arrangement is used in the Pallograph of O. Schlick, see Trans. Inst. Nav. Arch., Vol. 34, p. 167 (1893).

The same scheme is also used in the Vibrograph of the Cambridge Instrument Co. See Jr. Optical Soc. of America, Vol. 10, p. 455 (1925).

† For a more detailed consideration of this Vibrograph, see paragraph 9.

‡ This instrument is described by F. Lux, E. T. Z., 1905, p. 264–387.

the system of strips represents a definite series of frequencies. The difference between the frequencies of any two consecutive strips is usually equal to half a vibration per second.

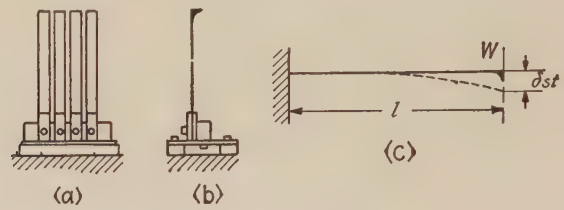


Fig. 12.

In figuring the frequency a strip can be considered as a cantilever beam (Fig. 12-*c*). In order to take into consideration the effect of the mass of the strip on the vibration it is necessary to imagine that one quarter of the weight W_1 of the strip is added * to the weight W, the latter being concentrated at the end. Then,

$$\delta_{st} = \frac{(W + W_1/4)l^3}{3EI}.$$

This statical deflection must be substituted in eq. (5) in order to obtain the period of natural vibration of the strip. In service the instrument is attached to the machine, the frequency of whose vibrations is to be measured. The strip whose period of natural vibration is nearest to the period of one revolution of the machine will be in a condition near resonance and a heavy vibration of this strip will be built up. From the frequency of the strip, which is known, the speed of the machine can be obtained. Instead of a series of strips of different lengths and having different masses at the ends, one strip can be used having an adjustable length. The frequency of vibration of the machine can then be found by adjusting the length of the strip in this instrument so as to obtain resonance. On this latter principle the well known Fullarton vibrometer is built.

c. Indicator of Steam Engines.—Steam engine indicators are used for measuring the variation of steam pressure in the engine cylinder. The accuracy of the records of such indicators will depend on the ability of the indicator system, consisting of piston, spring and pencil, to follow exactly the variation of the steam pressure. From the general discussion

* A more detailed consideration of the effect of the mass of the beam on the period of vibration is given in paragraph 14

of paragraph 3 it is known that this condition will be satisfied if the frequency of natural vibrations of the indicator system is very high in comparison with that of the steam pressure variation in the cylinder. Let A = .20 sq. in. = area of the indicator piston,

W = .133 lbs. = weight of the piston, piston rod and reduced weight of other parts connected with the piston,

s = .1 in. displacement of the pencil produced by the pressure of one atmosphere (15 lbs. per sq. in.),

n = 4 = the ratio of the displacement of the pencil to that of the piston.

From the condition that the pressure on the piston is equal to $15 \times .2 = 3.00$ lbs. produces a compression of the spring equal to $\frac{1}{4} \times .1 = .025$ in., we find that the scale of the spring is:

$$k = 3.00 : .025 = 120 \text{ lbs. in}^{-1}.$$

The frequency of the free vibrations of the indicator is (see eq. (5))

$$f = \frac{1}{\tau} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_{st}}} = \frac{1}{2\pi}\sqrt{\frac{gk}{W}} = \frac{1}{2\pi}\sqrt{\frac{386 \cdot 120}{.133}} = 94 \text{ per sec.}$$

This frequency can be considered as sufficiently high in comparison with the usual frequency of steam engines and the indicator's record of steam pressure will be sufficiently accurate. In the case of high speed engines, however, such an instrument may give completely unreliable records * under certain conditions.

d. Other Applications.—It is well known that inertia forces of counter weights in locomotive wheels produce additional pressure on the track. This effect of counterweights can easily be obtained by using the theory of forced vibrations. Let W = the weight of the wheel and of all parts rigidly connected to the wheel, Q = spring borne weight, P = centrifugal force due to unbalance, ω = angular velocity of the wheel. Considering then the problem as one of statics, the vertical pressure of the wheel on the rail (see Fig. 13) will be equal to

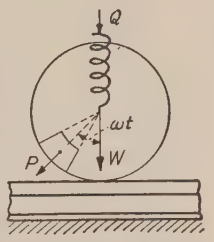


Fig. 13.

$$Q + W + P \cos \omega t. \qquad (a)$$

At slow speed this expression represents a good approximation for the

* The description of an indicator for high frequency engines (Collins Micro-Indicator) is given in Engineering, Vol. 113, p. 716 (1922). Symposium of Papers on Indicators, see Proc. Meetings of the Inst. Mech. Eng., London, Jan. (1923).

wheel pressure. In order to get this pressure with greater accuracy, forced vibrations of the wheel on the rail produced by the periodical vertical force $P \cos \omega t$ must be considered. Let k denote the vertical load on the rail necessary to produce the deflection of the rail equal to unity directly under the load and δ_{st}, the deflection produced by the weight W, then,

$$\delta_{st} = \frac{W}{k}.$$

The period of free vibrations of the wheel on the rail will be given by the equation * (see eq. (5)).

$$\tau = 2\pi\sqrt{\frac{W}{kg}}. \tag{b}$$

The period of one revolution of the wheel, i.e., the period of the disturbing force $P \cos \omega t$, is

$$\tau_1 = \frac{2\pi}{\omega}.$$

Now, by using eq. (20), it can be concluded that the dynamical deflection of the rail produced by the force P will be larger than the corresponding statical deflection in the ratio,

$$\frac{1}{1 - \left(\frac{\tau}{\tau_1}\right)^2}. \tag{c}$$

The pressure on the rail produced by the centrifugal force P will also increase in the same ratio and the maximum wheel pressure will be given by

$$Q + W + \frac{P}{1 - \left(\frac{\tau}{\tau_1}\right)^2}. \tag{d}$$

For a 100 lb. rail, a modulus of the elastic foundation equal to 1500 lbs. per sq. in. and $W = 6000$ lbs. we will have †

$$\tau = .068 \text{ sec.}$$

* In this calculation the mass of the rail is neglected and the compressive force Q in the spring is considered as constant. This latter assumption is justified by the fact that the period of vibration of the engine cab on its spring is usually very large in comparison with the period of vibration of the wheel on the rail, therefore vibrations of the wheel will not be transmitted to the cab and variations in the compression of the spring will be very small.

† See S. Timoshenko and J. M. Lessells, "Applied Elasticity," p. 334 (1925).

Assuming that the wheel performs five revolutions per sec. we obtain

$$\tau_1 = .2 \text{ sec.}$$

Substituting the values of τ and τ_1 in the expression (c) it can be concluded that the dynamical effect of the counterbalance will be about 11% larger than that calculated statically.

The *frequency of natural vibration of a system* can easily be obtained experimentally by applying to this system a disturbing force, the frequency of which can be gradually changed. In Fig. 14 the determination of the frequency of lateral vibrations of a rotor AB is shown. The periodical disturbing force is produced by a small variable speed motor with an eccentrically attached weight. The vertical component of the centrifugal force P of this weight represents the periodical disturbing force producing lateral vibrations of the rotor. These vibrations can be easily recorded by any kind of vibrograph. By gradually changing the speed of the motor the number of revolutions per minute of the motor at which the amplitude of the forced vibrations of the rotor becomes a maximum can be established. This number represents the frequency of the natural vibrations of the rotor. There are cases where a disturbing couple instead of a disturbing force must be applied to a system in order to produce forced vibrations. In such case a motor with two eccentrically attached weights, as shown in Fig. 14-b can be used.

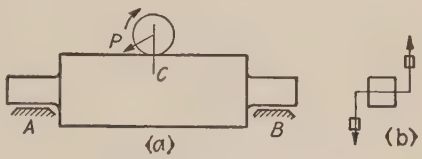


FIG. 14.

5. Another Method of Calculating Forced Vibrations.—From the general solution (19), p. 9, it is seen that at the moment of application of the disturbing force not only are forced vibrations of the system produced, but also free vibrations. The latter will be damped gradually due to various kinds of resisting forces but if the motion at the beginning of the vibration is being considered the free vibrations may be of importance. In order to obtain the complete displacement of the system at starting, a method based on solution (6), p. 3, for free vibrations will now be considered. It is seen from this solution that the displacement from the position of equilibrium due to any initial velocity $\dot{x}_0$ at any moment t is

$$x = \frac{\dot{x}_0}{p} \sin pt \qquad (a)$$

Assume now that a variable force is acting on the system shown in Fig. 1 and let q represent this disturbing force per unit mass (see eq. (17), p. 8). The variation of this force in some interval from $t = 0$ to $t = t_1$ can be represented graphically by a curve MN as shown in Fig. 15. In order to obtain the displacement of the system at any moment t_1 imagine the continuous action of the force divided into elementary intervals dt. Due to the action of the force q during this interval, the velocity of the weight W (Fig. 1) is increased a small amount, which can be found from the equation

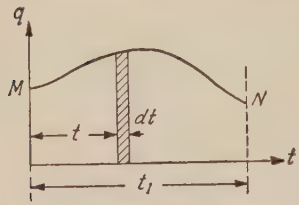


FIG. 15.

$$\frac{d\dot{x}}{dt} = q,$$

from which

$$d\dot{x} = q dt. \tag{b}$$

Now, from eq. (a) it can be concluded that any addition to the velocity at the time t, such as given by eq. (b), will produce at the moment t_1 (see Fig. 14) a displacement given by

$$dx = \frac{q dt}{p} \sin p(t_1 - t). \tag{c}$$

This expression represents the displacement due to the action of the force during an infinitely short interval dt only. In order to obtain the displacement produced by a continuous action of the force, the summation of such elementary displacements as (c) must be made. The displacement at the moment t_1 then becomes

$$x = \frac{1}{p} \int_0^{t_1} q \sin p(t_1 - t) dt. \tag{24}$$

This expression represents the complete displacement produced by the force q acting during the interval from $t = 0$ to $t = t_1$. It includes both forced and free vibrations and may become very useful in studying the motion of the system at starting. It can be used also in cases where an analytical expression for the disturbing force is not known and where the force q is given graphically or numerically. It is only necessary in such a case to determine the magnitude of the integral (24) by using one of the approximate methods of integration.*

* See von Sanden, "Practical Analysis," London, 1924.

As an example of the application of this method, the vibration under the action of a disturbing force $q = a \cos mt$ (see eq. (18), p. 9) will now be considered. Substituting this expression of q in eq. (24) we obtain,

$$x = \frac{a}{p}\int_0^{t_1} \cos mt \sin p(t_1 - t)dt = \frac{a}{p^2 - m^2}(\cos mt_1 - \cos pt_1).$$

This result coincides completely with the solution (19) obtained above (see p. 9). Equation (24) can be used also in cases where it is necessary to find the displacement of the load W (see Fig. 1) resulting from several impulses. Assume, for instance, that due to impulses obtained by the load W at the moments t^{I}, t^{II}, t^{III}, $\cdots$ increments of the speed $\Delta_1\dot{x}$, $\Delta_2\dot{x}$, $\Delta_3\dot{x}$, $\cdots$ be produced. Then from equations (b) and (24) the displacement at any moment t_1 will be,

$$x = \frac{1}{p}\left(\Delta_1\dot{x} \sin p(t_1 - t^{\text{I}}) + \Delta_2\dot{x} \sin p(t_1 - t^{\text{II}}) + \Delta_3\dot{x} \sin p(t_1 - t^{\text{III}}) + \cdots\right).$$

This displacement can be obtained very easily graphically by considering $\Delta_1\dot{x}$, $\Delta_2\dot{x}$, $\cdots$ as vectors inclined to the horizontal axis at angles $p(t_1 - t^{\text{I}})$, $p(t_1 - t^{\text{II}})$, $\cdots$ (Fig. 16). The vertical projection OC_1 of the geometrical sum OC of these vectors, divided by p, will then represent the displacement x given by the above equation. In cases where a constant force is applied at the moment $t = 0$ to the load W (Fig. 1) the displacement of the load at any moment t_1 becomes from eq. (24):

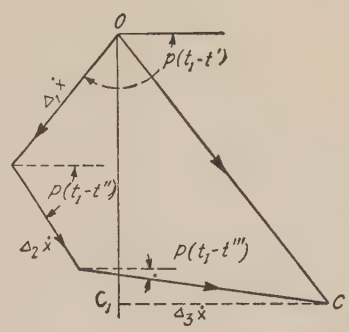


FIG. 16.

$$x = \frac{q}{p}\int_0^{t_1} \sin p(t_1 - t)dt = \frac{q}{p^2}(1 - \cos pt_1). \qquad (d)$$

q/p^2 = statical deflection due to the force q (see eq. (17)). It is seen, from (d), that the maximum deflection during vibrations produced by a suddenly applied force is equal to twice the statical deflection corresponding to the same force.

6. Damping Proportional to the Velocity.—In the previous consideration (see paragraph 1) all frictional resisting forces were neglected and, as a result, free vibrations of constant amplitude were obtained. In actual conditions, however, resisting forces although they may in some cases be small, cause a gradual damping of the original vibrations.

These *damping forces* may arise from several different sources, such as air or fluid friction, internal friction of the material of the vibrating body, or friction between sliding surfaces.

In calculating the friction between sliding surfaces the Coulomb-Morin law is usually applied.* It is assumed that in the case of dry surfaces the friction force is proportional to the normal component of the pressure acting between the surfaces and independent of the velocity of sliding. In the case of perfectly lubricated surfaces, when there exists a lubricating film between the sliding surface, the friction force is independent of the pressure and proportional to the velocity of sliding. In actual cases some intermediate condition, between the two extreme conditions mentioned above, usually exists. When a body is vibrating in air or in a liquid the resistance of the medium should be taken into consideration. In the case of very small velocities such as may exist in dash pots the resisting force is proportional to the velocity † while for larger velocities it may be assumed with sufficient accuracy to be proportional to the second power of the velocity.

When an elastic body is vibrating in a vacuum the vibration will be gradually damped out by internal friction in the material of the body.‡ The effect of this kind of friction on the vibration can be taken into account by assuming a resisting force proportional to the velocity provided the factor of proportionality is determined from experiments. The resisting force is also proportional to the velocity in the case of electrical damping.§

The simplest case will now be considered in which the damping is proportional to the velocity.|| We take the same arrangement as before

* C. A. Coulomb: Theorie des machines simples, Paris, 1821. A. Morin: Mém. prés. p. div. sav., Vol. 4 (Paris, 1833) and Vol. 6 (1835). For a review of the literature on friction, see R. v. Mises, Encyklopädie d. Math. Wissenschaften, Vol. 4, p. 153. For references to new literature on the same subject, see G. Sachs, Z. f. Angew. Math. und Mech., Vol. 4 (1924), p. 1, and H. Fromm, Z. f. Angew. Math. und Mech., Vol. 7 (1927), p. 27.

† See experiments by A. Stodola, Schweiz. Bauzeitung, Vol. 22, p. 113 (1893).

‡ The data on the amount of this internal friction for various materials can be found in a book by Dr. E. Lehr, "Die Abkurzungsverfahren zur Ermittlung der Schwingungsfestigkeit von Materialen," Dissertation, Stuttgart, 1925. See also A. L. Kimball, Mechanical Engineering May 1927, p. 440 and the discussion to Ormondroyd and Den Hartog's paper Trans. A.S.M.E. Appl. Mech. Div. 1928.

§ Some data on this kind of damping and on some other kinds of damping can be found in a book by H. Holzer: "Die Berechnung der Drehschwingungen," Berlin, 1921, p. 92.

|| Damping proportional to the square of velocity is considered by W. Hort, see Technische Schwingungslehre, 2d Ed., p. 44 (1922).

(see Fig. 1) and add the friction force $-\alpha\dot{x}$ to the forces acting on the weight W during vibration. Here α is a constant depending on the conditions and the minus sign indicates that the force is in the direction opposite to the velocity. The following equation of motion (see eq. (1), p. 2) will be obtained:

$$W - \frac{W}{g}\ddot{x} - \alpha\dot{x} = W + kx, \tag{a}$$

α = damping

from which

$$\ddot{x} + 2n\dot{x} + p^2x = 0, \tag{25}$$

in which

$$p^2 = \frac{kg}{W} \quad \text{and} \quad 2n = \frac{\alpha g}{W}. \tag{26}$$

$\omega_N = p$

By using the notation,

$$p_1^2 = p^2 - n^2, \tag{27}$$

the general solution of eq. (25) can be written in the following form: *

$$x = e^{-nt}(A \sin p_1t + B \cos p_1t). \tag{b}$$

It is seen that the vibratory motion obtained has a period

$$\tau = \frac{2\pi}{p_1} = \frac{2\pi}{\sqrt{p^2 - n^2}}. \tag{28}$$

If the quantity n is small in comparison with p, the difference between p_1 and p is a small quantity of the second order (see eq. (27)), and therefore it can be assumed with sufficient accuracy that a small damping force proportional to the speed does not affect the period of vibration.

The amplitude of the vibrations, given by eq. (*b*), due to the factor e^{-nt}, gradually decreases with the time and the vibrations originally generated will be gradually damped. In order to obtain this amplitude the arbitrary constants A and B of solution (*b*) must be determined from the initial conditions. Let x_0 and $\dot{x}_0$ denote, respectively, the displacement of the system from the position of equilibrium and the velocity at the initial moment $t = 0$, then, from equation (*b*), we have

$$x_0 = B.$$

Differentiating the same equation with respect to time and using the magnitude of the initial velocity $\dot{x}_0$ we obtain

$$A = \frac{1}{p_1}(\dot{x}_0 + nx_0)$$

* It is assumed here that the quantity n, depending on the friction, is smaller than p, so that $\sqrt{p^2 - n^2}$ is real; otherwise *aperiodic* motion takes place.

and solution (*b*) becomes

$$x = e^{-nt}\left(\frac{\dot{x}_0}{p_1}\sin p_1 t + x_0(\cos p_1 t + \frac{n}{p_1}\sin p_1 t)\right) \cdot \tag{29}$$

Solution (29) can be represented graphically. For instance, for a particular case when

$$\frac{\dot{x}_0}{p_1} + \frac{nx_0}{p_1} = 0,$$

i.e., when

$$x = x_0 e^{-nt}\cos p_1 t \tag{c}$$

we will have the curve as shown in Fig. 17. This curve is tangent to the curve $x = x_0 e^{-nt}$ in the points m_1, m_2, m_3, where $t = 0$, $t = \tau$, $t = 2\tau$, $\cdots$; and to the curve $x = -x_0 e^{-nt}$ in the points m_1^1, m_2^1, $\cdots$ where $t = 1/2\tau$, $t = 3/2\tau$, $\cdots$. These points do not coincide with the points of extreme displacements of the system from the position of equilibrium and it is easy to see that due to damping, the time interval necessary for

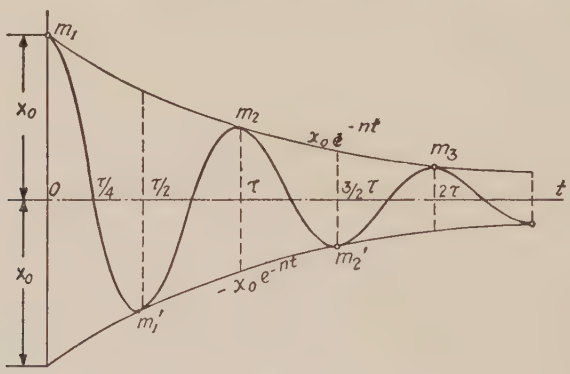


FIG. 17.

displacement of the system from a middle position to the subsequent extreme position is less than that necessary to return from an extreme position to the subsequent middle position.

The rate of damping depends on the magnitude of the constant n (see eq. (25)). It is seen from the general solution (29) that the amplitude of the vibration diminishes after every cycle in the ratio

$$e^{-n\tau} : 1, \tag{d}$$

i.e., it decreases following the law of geometrical progression. Equation (*d*) can be used for an experimental determination of the coefficient of

damping n. It is only necessary to determine by experiment in what proportion the amplitude of vibration is diminished after a given number of cycles.

7. Constant Damping.—As an example of constant damping, the arrangement shown in Fig. 18 will be considered. Assume that the weight W is brought in its extreme right position $x = x_0$ and released; then motion towards the left begins. The following forces are acting on the load during this motion: (1) a tensile force in the spring equal to kx;* (2) an inertia force—$W/g\ddot{x}$; and (3) a constant friction R between the load and the sliding rod, in a direction opposite to that of motion. The equation of motion will be

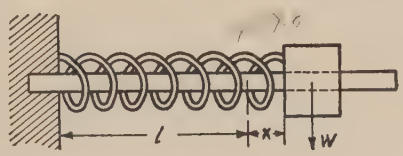


FIG. 18.

$$-kx - \frac{W}{g}\ddot{x} \mp R = 0 \qquad (a)$$

or

$$\ddot{x} + p^2(x - a) = 0; \qquad (30)$$

here, as before (eq. 3)

$$p = \sqrt{\frac{kg}{W}}$$

and

$$a = \frac{R}{k}. \qquad (31)$$

This latter quantity has a simple physical meaning. It is the extension of the spring produced by the frictional force R. Making $x_1 = x - a$ equation (30) reduces to

$$\ddot{x}_1 + p^2 x_1 = 0.$$

This equation is in complete agreement with the equation (2) (p. 2) previously considered, and the following conclusions may be drawn:

(1) A constant friction does not change the period of vibration;

(2) The middle position about which the simple harmonic vibration takes place is displaced in the direction of the force R of friction by the distance a (eq. 31). Due to this latter fact, the extreme left position of the load will be distant only $x_0 - 2a$ from the position corresponding to the unstrained condition of the spring. This means that a diminishing

* x is measured from the position of unstrained condition of the spring.

of the amplitude produced by constant damping during half a cycle is equal to $2a$.

Considering now the motion from the extreme left position to the right, and applying the same reasoning, it can be shown that during the second half of the cycle a further diminishing of the amplitude by the quantity $2a$ will be produced. Therefore it is seen that in this case considered the decrease of the amplitude follows the law of arithmetical progression. The load W will remain in one of its extreme positions as soon as the amplitude becomes less than a. It is clear that in this position the friction will be sufficient to balance the tensile force of the spring.

8. Forced Vibrations with Damping.—Taking again the arrangement shown in Fig. 1, assume that, in addition to the forces previously considered * (see eq. (a), p. 2), a periodic force $Q \sin mt$ is acting in a vertical direction on the load W. Then using the notation

$$\frac{gQ}{W} = q, \tag{a}$$

the equation of motion becomes †

$$\ddot{x} + 2n\dot{x} + p^2x = q \sin mt. \tag{32}$$

A general solution of this equation will be obtained if, to the solution of the corresponding homogeneous equation (see eq. (25), p. 24) a particular solution of eq. (32) will be added. The latter will have the form

$$x = M \sin mt + N \cos mt.$$

Substituting this in eq. (32) the following equations for determining the constants M and N will be obtained:

$$\begin{aligned} -Nm^2 + 2Mmn + Np^2 &= 0, \\ -Mm^2 - 2Nmn + Mp^2 &= q, \end{aligned}$$

from which

$$N = -\frac{2qmn}{(p^2 - m^2)^2 + 4m^2n^2}, \qquad M = \frac{q(p^2 - m^2)}{(p^2 - m^2)^2 + 4m^2n^2}.$$

Now the general solution of eq. (32) becomes †

$$x = e^{-nt}(A \sin p_1t + B \cos p_1t) - \frac{2qmn}{(p^2 - m^2)^2 + 4m^2n^2} \cos mt + \frac{q(p^2 - m^2)}{(p^2 - m^2)^2 + 4m^2n^2} \sin mt. \tag{33}$$

* The case of forced vibration with constant friction is discussed in the paper by W. Eckolt, Zeitschrift f. Tech. Phys., Vol. 7, 1926, p. 226.

† The notations of paragraph 6 are used here.

The first member on the right side having the factor e^{-nt} represents the damped vibrations studied in the previous paragraph. The two other terms, proportional to q, represent the forced vibrations. They are maintained by the periodical external force and will not be damped with the time. Their period $\tau_1 = 2\pi/m$ is the same as that of the disturbing force and their amplitude is proportional to the magnitude of this force. This amplitude depends also on the relation between the period τ of the natural vibration of the system and the period τ_1 of the disturbing force.

The expression for the forced vibration can be simplified by using the following notations:

$$\frac{-2qmn}{(p^2 - m^2)^2 + 4m^2n^2} = -C \sin \alpha, \tag{b}$$

$$\frac{q(p^2 - m^2)}{(p^2 - m^2)^2 + 4m^2n^2} = C \cos \alpha. \tag{c}$$

The expression for the forced vibrations becomes

$$C(\cos \alpha \sin mt - \sin \alpha \cos mt) = C \sin (mt - \alpha). \tag{34}$$

The amplitude C of forced vibrations, from (b) and (c) is

$$C = \frac{q}{\sqrt{(p^2 - m^2)^2 + 4m^2n^2}}. \tag{35}$$

The angle α in eq. (34) represents the *difference in phase* between the forced vibration and the disturbing force.

Dividing (b) by (c) we have

$$\tan \alpha = \frac{2mn}{p^2 - m^2}. \tag{36}$$

If $p > m$, as occurs when the period of the natural vibration of the system is less than that of the disturbing force, α becomes positive and less than $\pi/2$. It can be concluded from eq. (34) that the forced vibration lags behind the disturbing force $q \sin mt$.

When $p < m$,

$$\frac{\pi}{2} < \alpha < \pi,$$

i.e., the lag of forced vibration is greater than $\pi/2$.

When $p = m$, i.e., in the case of resonance, $\tan \alpha$ in eq. (36) becomes infinite and the difference in phase becomes equal to $\pi/2$. This

means that during such vibratory motion the system assumes its middle position at the moment when the disturbing force attains its maximum value.

The amplitude C of this forced vibration will now be considered. Remembering that

$$q = \frac{gQ}{W}$$

and

$$p^2 = \frac{kg}{W},$$

it will be concluded that

$$\frac{q}{p^2} = \frac{Q}{k} = \delta_{st}, \tag{d}$$

where δ_{st} denotes the deflection which would be produced by the maximum disturbing force if it were to be applied statically.

By using (d) the expression of the amplitude of the forced vibrations (eq. 35) becomes

$$C = \frac{\delta_{st}}{\sqrt{\left(1 - \frac{m^2}{p^2}\right)^2 + \frac{4m^2n^2}{p^4}}} = \frac{\delta_{st}}{\sqrt{\left(1 - \frac{\tau^2}{\tau_1^2}\right)^2 + \frac{\tau^2\gamma^2}{\tau_1^2}}}, \tag{37}$$

in which $\tau = 2\pi/p$ = period of the natural vibration of the system,

$\tau_1 = 2\pi/m$ = period of the disturbing force $Q \sin mt$,

$\gamma = 2n/p$ = quantity depending on the magnitude of the damping forces.

It is easy to see that if the disturbing force has a very large period τ_1, C approaches the value δ_{st}. This means that in such cases the deflections of the system can be calculated on the assumption that the disturbing force $Q \sin mt$ is acting statically. Another extreme case will be obtained on the assumption that τ_1 is a very small quantity, as occurs when the disturbing force has a very high frequency. In such case the denominator of equation 37 becomes very large and the amplitude of the forced vibrations approaches zero.

If τ_1 approaches τ and the damping forces are small, the denominator in expression 37 becomes very small. This means that in the case of resonance a comparatively small disturbing force may produce very large forced vibrations, and very serious stresses in a vibrating structure.

In order to give a clear picture of the variation of the amplitude C with the variation of the ratio τ/τ_1 and of the quantity γ, several curves

representing the magnification factor

$$\beta = \frac{1}{\sqrt{\left(1 - \frac{\tau^2}{\tau_1^2}\right)^2 + \frac{\tau^2\gamma^2}{\tau_1^2}}} \tag{38}$$

as a function of the ratio τ/τ_1 are plotted in Fig. 19 for different values of the quantity γ.* It is seen from these curves that the points of maximum amplitude do not coincide exactly with the condition of resonance, but that the magnitude of the ratio τ/τ_1, corresponding to the maximum

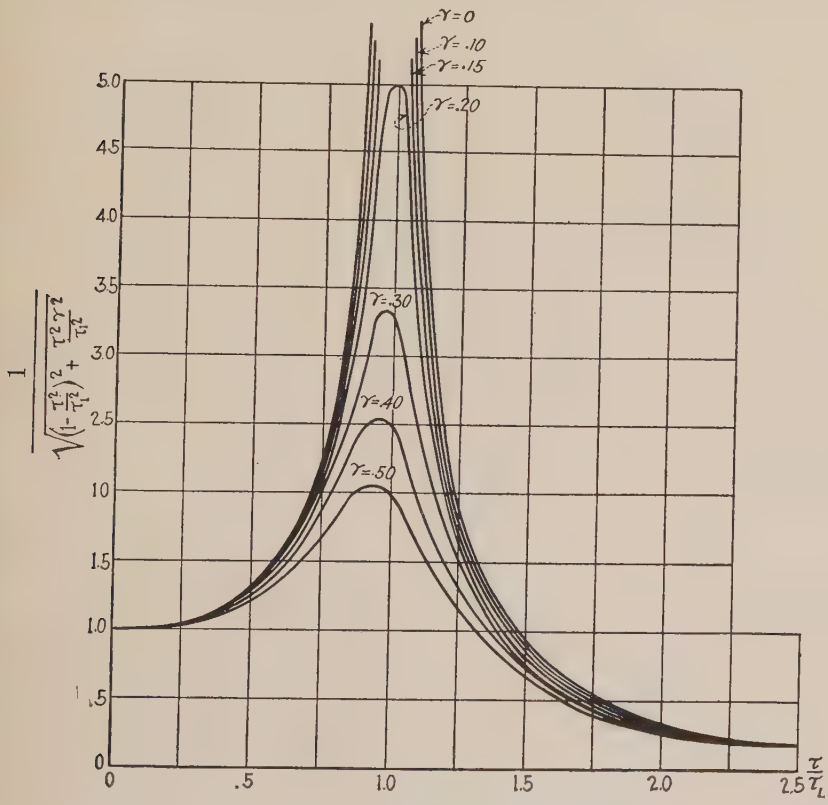


FIG. 19.

amplitude is somewhat less than unity and approaches this limit with a decrease of damping. Therefore in the case of a small damping the

* For $\gamma = 0$ eq. (38) coincides completely with the eq. (20), p. 9 obtained for vibrations without damping.

difference between the maximum amplitude of the forced vibration and the amplitude of this vibration at resonance is very small, and the latter amplitude is customarily used as a basis for calculating the maximum stress during forced vibration.

From Fig. 19 it is easy to see that damping has a large effect on forced vibrations only in the region near the resonance condition, say from $\tau/\tau_1 = .75$ to $\tau/\tau_1 = 1.25$. In this range damping considerably reduces the amplitude of the forced vibrations and keeps it finite at resonance.

With increasing damping the resonance effect becomes less and less pronounced. Outside of this interval small variations in damping produce only a secondary effect on the amplitude of the vibrations and the conclusions previously made (see paragraph 3) for the extreme high and low frequencies of the disturbing force remain unchanged by damping.

It has been shown that the forced vibration always lags behind the disturbing force and that the difference in phase α can be determined from eq. (36). This phase difference may have some practical interest and its variation with the damping and with changing ratio τ/τ_1 is shown in Fig. 20. This shows that in the region near to resonance a very sharp

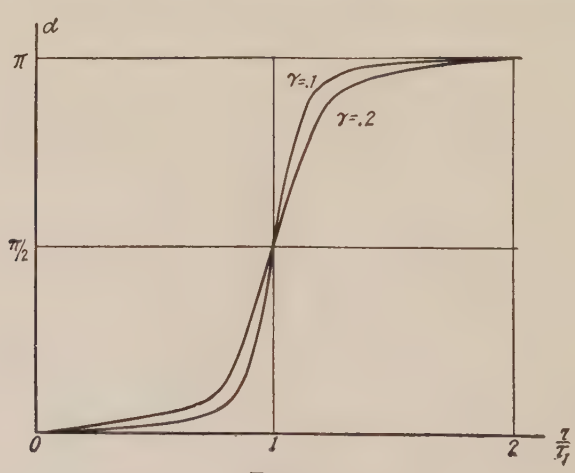


Fig. 20.

variation in the phase of the forced vibrations takes place if the damping is small. For $\tau/\tau_1 < 1$ the difference of phase α is small but becomes equal to $\pi/2$ for the case $\tau = \tau_1$ and approaches π when $\tau/\tau_1 > 1$. By a gradual diminishing of γ the condition considered in paragraph 3 (see p. 11) will be approached, where the difference in phase suddenly changes from 0 to π.

From eq. (34) it is seen that the maximum velocity in forced vibrations is equal to Cm, therefore the maximum kinetic energy of the system will be proportional to

$$C^2m^2 = \frac{\delta_{st}^2 m^2}{\left(1 - \frac{m^2}{p^2}\right)^2 + \frac{4m^2n^2}{p^4}} = \frac{\delta_{st}^2 p^2}{\left(\frac{p}{m} - \frac{m}{p}\right)^2 + \frac{4n^2}{p^2}}. \qquad (e)$$

Since this expression has its maximum value when $m = p$, the kinetic energy of the vibrating system will become maximum under the condition of resonance.

Consider now the rate at which work is being done in maintaining the vibration against resistance. The impressed force per unit mass at any moment t is equal to $q \sin mt$ (see eq. (32)). The velocity of the point of application of the force at the same moment will be, from eq. (34),

$$\dot{x} = Cm \cos (mt - \alpha).$$

The rate of work produced by the impressed force then becomes

$$q \sin mt \cdot Cm \cos (mt - \alpha) = \frac{qmC}{2} \{\sin (2mt - \alpha) + \sin \alpha\}. \qquad (f)$$

In calculating the average value of the quantity (f) during one cycle, the first term in the bracket on the right side of equation (f) will give us 0 and we obtain for this average value w the following result:

$$w = \frac{qmC \sin \alpha}{2},$$

or, by eq. (b)

$$w = \frac{q^2m^2n}{(p^2 - m^2)^2 + 4m^2n^2} = \frac{q^2}{2p} \frac{\frac{2n}{p}}{\left(\frac{p}{m} - \frac{m}{p}\right)^2 + \left(\frac{2n}{p}\right)^2}.$$

Using the notations,

$$\frac{2n}{p} = \gamma,$$

$$\frac{p}{m} = \frac{\tau_1}{\tau} = 1 + z,$$

we obtain

$$w = \frac{q^2}{2p} \frac{\gamma}{\left(1 + z - \frac{1}{1+z}\right)^2 + \gamma^2}. \qquad (39)$$

The quantity w attains its maximum value at resonance. Substituting

$z = 0$ in eq. (39) we have

$$w_{\max} = \frac{q^2}{2p\gamma} = \frac{q^2}{4n}. \tag{40}$$

It is seen that $w_{\max}$ increases with a decrease of the quantity n, i.e., with a decrease in damping.* In studying the variation of w (see eq. (39)) near the point of resonance, z can be considered as a small quantity. Then

$$\left(1 + z - \frac{1}{1+z}\right)^2 \approx 4z^2$$

and we have, from eq. (39),

$$w = \frac{q^2}{2p}\,\frac{\gamma}{4z^2 + \gamma^2}. \tag{39^1}$$

The variation of the factor

$$\frac{\gamma}{4z^2 + \gamma^2}$$

near the point of resonance is shown in Fig. 21. It is seen that the

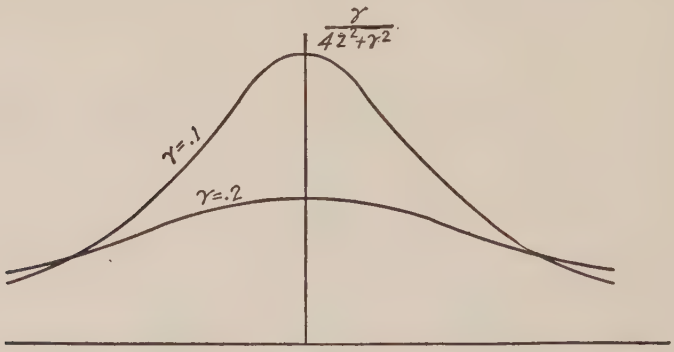


Fig. 21.

variation of w becomes more and more marked with a decrease in damping. In Fig. 22 the work w is plotted against the ratio τ/τ_1 for $\gamma = .1$ and $\gamma = .5$. It is seen that only near to the point of resonance the absorption of energy increases with decreasing damping. In other points a greater absorption of energy corresponds to a greater damping.

This general conclusion may be of some interest in discussing the amount of energy absorbed in machinery foundations due to vibration.

* In the case $n = 0$ the amplitude of vibration increases indefinitely and the reasoning given above cannot be applied.

9. Technical Applications (Forced Vibrations with Damping).—Several cases where the previously discussed theory on forced vibrations can be applied will now be considered.

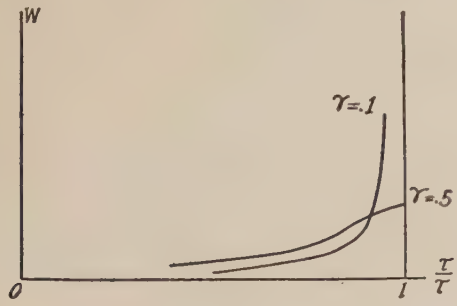


FIG. 22.

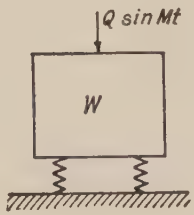


FIG. 23.

a. Vibration Absorbers.—In order to reduce the effect of disturbing forces on the foundation of a machine and to prevent the transmission of vibrations to the building, the introduction of special flexible elements between the machine and the foundation may be useful. Let W = the weight of the machine, $Q \sin mt$ = the impressed periodical force. If the machine is rigidly attached to the foundation the entire disturbing force will be transmitted to the foundation. By introducing the flexible springs of a vibration absorber between the machine and the foundation a substantial reduction in the force transmitted to the support can be obtained provided certain conditions are fulfilled. The calculation of the effect of an absorber can easily be made on the basis of the previously developed theory of forced vibration (see paragraph 8). By choosing the springs of the absorber in such a manner as to make the period τ of the natural vibration of the machine W on the springs large in comparison with the period $\tau_1 = 2\pi/m$ of the impressed force, the dynamical displacements of the machine during vibration will be reduced in the ratio β (see eq. (38)), as compared with the displacements which would be produced by the disturbing force $Q \sin mt$ under static conditions. The forces, which are transmitted to the foundation, are proportional to the deflection of the springs and therefore will be reduced in the ratio β also. Neglecting damping and taking, for instance, $\tau/\tau_1 = 4$, i.e., assuming that the operating speed is four times that corresponding to resonance, we have, from eq. (38), $\beta = 1/15$, i.e., due to the effect of the vibration absorber the force transmitted to the foundation will be reduced to 1/15 of the impressed force. The actual design of an absorber frequently

offers great difficulties, principally on account of space limitation. It is difficult to have sufficient space for placing springs which will be strong enough to transmit the load to the foundation and at the same time flexible enough to make the resonance speed of the system small in comparison with the operating speed.

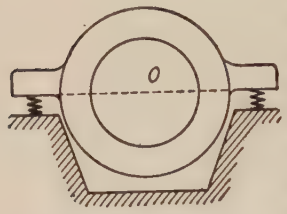


Fig. 24.

In Fig. 24, a schematic view of a vibration absorber for a large single-phase generator is shown.* In order to reduce the effect of the pulsating torque flexible springs are introduced between the stator of the machine and the foundation. By the use of special guides the possibility of lateral displacement of the stator is prevented and only a rotation about the O axis remains possible.

Let

k = torque moment necessary to produce a rotation of one radian of the stator about the axis going through O and perpendicular to the plane of the figure.

I = moment of inertia of the stator about the same axis.

φ = the angle of rotation of the stator during vibration.

The differential equation of vibration of the stator will then coincide with eq. (11) (see p. 6) and the period τ of vibration will be found from eq. (12). Neglecting damping, the effect of the vibration absorber will be represented by a magnification factor (see eq. 20)

$$\beta = \frac{1}{1 - \frac{\tau^2}{\tau_1^2}},$$

in which τ_1 = period of pulsating torque acting on the stator. If τ is large in comparison with τ_1, only a small part of the torque will be transmitted through the springs into the foundation.

An absorber consisting of steel springs has been considered above. In many cases, absorbers consisting of pads made from different kinds of organic materials such as rubber, cork, etc., are used for preventing transmission of vibrations to the building. It must be noted that such absorbers are usually too stiff to reduce substantially the effect of the principal harmonic of the impressed force but may be useful in absorbing

* See C. R. Soderberg, Electric Journal, Vol. XXI, p. 160 (1924).

vibrations due to higher harmonics of the disturbing forces. Moreover, the properties of these organic materials usually change with time. This is the reason why in all more important cases, steel spring absorbers must be used since they are more reliable.

b. Vibrograph.—Consider now as a further example, the vibrograph as shown in Fig. 11, of which the frame is subjected to a simple harmonic vibration. Let x denote the vertical displacement of the weight W from the position of equilibrium and $x_1 = b \sin mt$—the simple harmonic motion in a vertical direction of the frame of the vibrograph. Then the inertia force of the load W is $-W\ddot{x}/g$, the force in the spring $-k(x - x_1)(a/l)$ and the damping force acting on the weight W is $-c\dot{x}$. By neglecting the mass of the lever BD and taking the moment of all the forces about the knife edge B, we have

$$-\frac{W}{g}\ddot{x}l - c\dot{x}l - k(x - x_1)\frac{a^2}{l} = 0$$

or

$$\ddot{x} + 2n\dot{x} + p^2x = q \sin mt. \tag{a}$$

Here,

$$\frac{cg}{W} = 2n; \qquad \frac{kga^2}{Wl^2} = p^2; \qquad \frac{kbga^2}{Wl^2} = q. \tag{b}$$

The general solution of eq. (a) has been already obtained (see eq. (33), p. 26). The corresponding forced vibrations are (see eq. (34) and (37), p. 27)

$$x = \frac{b \sin (mt - \alpha)}{\sqrt{\left(1 - \frac{m^2}{p^2}\right)^2 + \frac{4m^2n^2}{p^4}}} = \beta b \sin (mt - \alpha), \tag{c}$$

where β denotes the factor of magnification. The recording instrument usually gives the displacement $x - x_1$ of the weight W with respect to the frame of the instrument. By using eq. (c) we have

$$\begin{aligned} x - x_1 &= b\{\beta \sin (mt - \alpha) - \sin mt\} \\ &= b\{(\beta \cos \alpha - 1) \sin mt - \beta \sin \alpha \cos mt\}. \end{aligned}$$

This represents a simple harmonic motion of the amplitude (see eq. (8))

$$A = b\sqrt{\beta^2 - 2\beta \cos \alpha + 1}. \tag{d}$$

It is seen that the scale of the record depends not only on the magnification factor β but also on the magnitude of the lag α. If damping can be

neglected, $\alpha = 0$ or $\alpha = \pi$ (see page 11) and eq. (d) yields:

$$A = b(\beta \pm 1). \qquad (d^1)$$

The minus sign corresponds to the case when $p > m$, i.e., when the frequency of the impressed force is less than that of natural vibrations of the vibrograph. The plus sign corresponds to the case $p < m$. The magnitude of the amplitude can be obtained for both cases from Fig. 7, by displacing the horizontal axis as shown by the dotted line.

Substituting in (d^1) the value of β from eq. (c) and neglecting damping, we have,

$$A = \frac{b}{\frac{p^2}{m^2} - 1}, \qquad (d^{11})$$

where the sign is valid for all values of m. Now, if the frequency p of the natural vibrations of the instrument is small, compared with m, we find,

$$A = -b \text{ approximately.}$$

Therefore, the amplitude of the record is equal to the amplitude of the impressed motion. Therefore, if we want to use the instrument as a *vibrograph* (to record *motions*) it is necessary to have $p \ll m$.

If, however, the opposite condition exists, ($p \gg m$), equation (d^{11}) reduces to:

$$A = b\frac{m^2}{p^2},$$

and since in a given instrument p is a constant, the recorded amplitude is proportional to bm^2. The impressed motion is $b \sin mt$: its acceleration is $-bm^2 \sin mt$, which is also proportional to bm^2. Therefore, if an instrument is made where $p \gg m$, it can be used as *accelerometer*.

Near the condition of resonance ($p = m$) a sharp variation in the magnitude of the lag α takes place and the factor β depends to a large extent on the magnitude of damping. This means that the determination of the amplitude A from eq. (d) becomes uncertain near this point and the record of the vibrograph in this region cannot give us reliable data.

*c. Sommerfeld's Experiment.**—This experiment shows very clearly how much energy can be dissipated and absorbed by the foundation when the machine is working at a condition of resonance. In his experiments. Sommerfeld attached a small d.c. motor to the top of a table. The motor

* A. Sommerfeld, Zeitsch. d. Ver. D. Ing., 1904.

had an eccentrically attached weight on its axis so that during rotation a disturbing force was produced which acted on the table. By gradually increasing the speed of the motor it was easily ascertained that at 310 r.p.m. a large horizontal oscillatory motion of the top of the table took place. Evidently this marked the condition of resonance for the system consisting of the table top and legs. In these vibrations the table board behaved as a rigid body and the horizontal motion was due to the bending of the legs alone. In having such a condition of resonance and by gradually increasing the voltage of the motor it can be clearly shown that the motor will have the tendency to maintain a constant speed corresponding to resonance ($n = 310$ in this case) and that the additional energy produces only further increase in forced vibration (see Fig. 25)

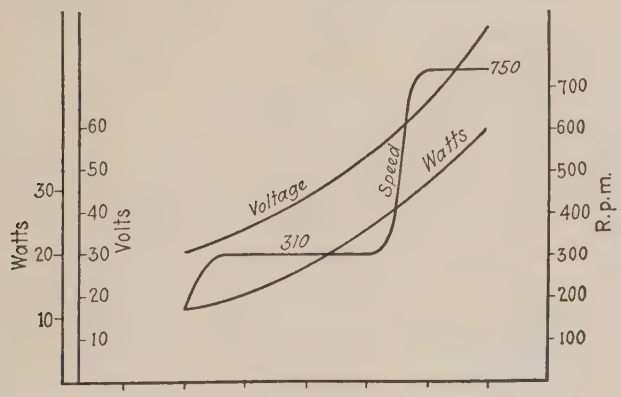


Fig. 25.

which is absorbed in the foundation. It is only after the voltage applied to the motor has been considerably increased that the motor is able to increase its speed beyond the critical range after which the forced vibrations quickly die out.

A second critical speed, in the case of Sommerfeld's experiment, was obtained at 750 r.p.m. At this speed the flexural vibrations of the table board came into resonance and large bending vibrations of the table top took place. Again the conditions became such that the foundation began to absorb a considerable amount of energy, which energy was dissipated in maintaining the forced vibration of the system.*

It is interesting to mention here that when the system described above is in the condition of resonance and a large amount of energy is

* See previous discussion, pp. 31 and 32.

transmitted to the foundation a sudden increase in speed of the motor can be obtained without changing the voltage by preventing by some kind of constraint heavy forced vibrations of the foundation. The writer carried out this experiment by using a small motor attached to a flexible steel beam on two supports. When the system was in resonance heavy flexural vibrations of the beam took place. By pressing the hand on the beam these vibrations were prevented and a sudden increase in the speed of the motor was easily demonstrated.

It must be noted here that the problem on forced vibrations of foundations is of great practical importance and becomes more and more urgent with increase in the capacity of large turbine generator units. In general this problem is much more complicated than is described above and some further aspects of it will be considered later in studying systems having several degrees of freedom.

10. Balancing of Rotating Machines.—One of the most important applications of the theory of vibrations is in the solution of balancing problems. It is known that a rotating body does not exert any variable disturbing action on the supports when the axis of rotation coincides with one of the principal axes of inertia of the body. It is difficult to satisfy this condition exactly in the process of manufacturing because due to errors in geometrical dimensions and non-homogeneity of the material some irregularities in the mass distribution are always present. As a result of this variable disturbing forces occur which produce vibrations. In order to remove these vibrations in machines and establish quiet running conditions, balancing becomes necessary. The importance of balancing becomes especially great in the case of high speed machines. In such cases the slightest unbalance may produce a very large disturbing force. For instance, at 1800 r.p.m. an unbalance equal to one pound at a radius of 30 inches produces a disturbing force equal to 2760 lbs.

In order to explain the various conditions of unbalance a rotor shown in Fig. 26, *a* will now be considered.* Imagine the rotating body divided into two parts by any cross section *mn*. The three following typical cases of unbalance may arise:

1. The centers of gravity of both parts may be in the same axial plane and on the same side of the axis of rotation as shown in Fig. 26, *b*. The center of gravity C of the whole body will consequently be in the same plane at a certain distance from the axis of rotation. This is called "*static unbalance*," because it can be detected by a statical test. A

* The rotor is considered as an absolutely rigid body and vibrations due to elastic deflections of it are neglected.

statical balancing test consists of putting the rotor with the two ends of its shaft on absolutely horizontal, parallel rails. If the center of gravity of the whole rotor is in the axis (Fig. 26, *c*) the rotor will be in static

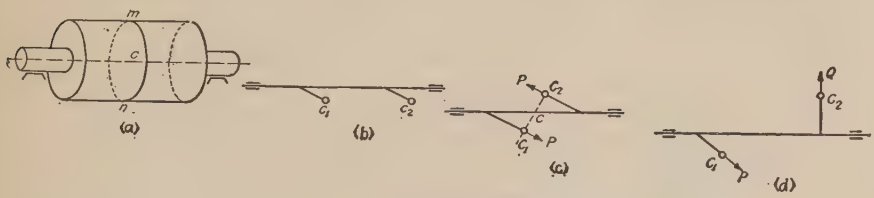


Fig. 26.

equilibrium in any position; if the center is slightly off the shaft, as in Fig. 26, *b*, it will roll on the rails till the center of gravity reaches its lowest position.

2. The centers of gravity of both parts may be in the same axial plane but on opposite sides of the axis of rotation as shown in Fig. 26, *c* and at such radial distances that the center of gravity C of the whole body will be exactly on the axis of rotation. In this case the body will be in balance under static conditions, but during rotation a disturbing couple of centrifugal forces P will act on the rotor. This couple rotates with the body and produces vibrations in the foundation. Such a case is called "*dynamic unbalance.*"

3. In the most general case the centers of gravity, C_1 and C_2, may lie in different axial sections and during rotation a system of two forces formed by the centrifugal forces P and Q will act on the body (see Fig. 26, *d*). This system of forces can always be reduced to a couple acting in an axial section and a radial force, i.e., static and dynamic unbalance will occur together.

It can be shown that in all cases complete balancing can be obtained by attaching to the rotor a weight in each of two cross sectional planes arbitrarily chosen. Consider, for instance, the case shown in Fig. 27. Due to unbalance two centrifugal forces P and Q act on the rotor during motion. Assume now that the weights necessary for balance must be located in the cross sectional planes I and II. The centrifugal force P can be balanced by two forces P_1 and P_2, lying with P in the same axial section. The magnitude of these forces will be determined from the following equations of statics

$$P_1 + P_2 = P,$$
$$P_1 a = P_2 b.$$

In the same manner the force Q can be balanced by the forces Q_1 and Q_2. The resultant of P_1 and Q_1 in plane I, and the resultant of P_2 and Q_2 in plane II will then determine the magnitudes and the positions of the correction weights necessary for complete balancing of the rotor.

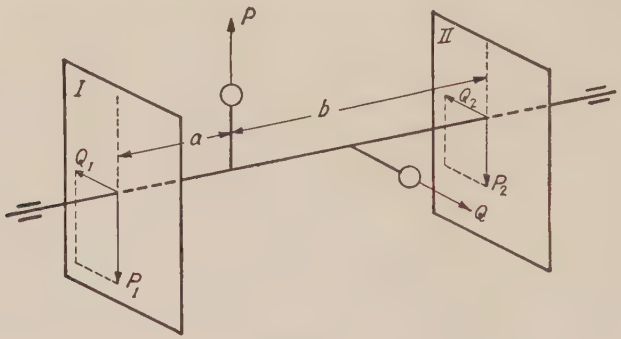


Fig. 27.

It is seen from this discussion that balancing can be made without any difficulty if the position and magnitude of the unbalance is known. For determining this unbalance various types of balancing machines are used and the fundamentals of these machines will now be discussed.

11. Machines for Balancing.—A balancing machine represents usually an arrangement in which the effects of any unbalance in the rotor which is under test may be magnified by resonance. There are two principal types of balancing machines: (1) machines where the location and the magnitude of unbalance is determined from records of the vibrations as in the machine of Lawaczeck-Heymann and (2) machines in which the same quantities are determined by means of compensating weights as in the machine of Akimoff.

The *machine* of *Lawaczeck-Heymann* consists mainly of two independent pedestals. The two bearings supporting the rotor are attached to springs which allow vibrations of the ends of the rotor in a horizontal axial plane. One of the bearings is locked with the balancing being performed on the other end (see Fig. 28). Any unbalance will produce vibration of the rotor about the locked bearing as a fulcrum. In order to magnify these vibrations all records are taken at *resonance conditions*. By a special motor the rotor is brought to a speed above the *critical* and then the motor power is shut off. Due to friction the speed of the rotor gradually decreases and as it passes through its *critical value* pronounced forced vibrations of the unlocked bearing of the rotor will

be produced by any unbalance. The process of balancing then consists of removing these vibrations by attaching suitable correction weights. The most suitable planes for placing these weights are the ends of the rotor body, where usually special holes for such weights are provided

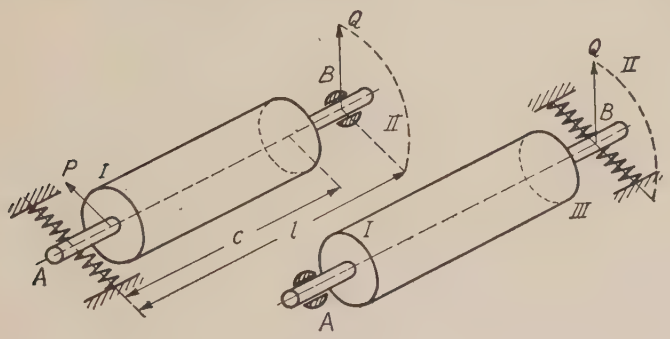


Fig. 28.

along the circumference. By such an arrangement the largest distance between the correction weights is obtained; therefore the magnitude of these weights is brought to a minimum. When the plane for such correction weight has been chosen there still remain two questions to be answered:

1. The location of the correction weight and
2. Its magnitude.

Both these questions can be solved by trial. In order to determine the location, some arbitrary correction weight should be put in the plane of balancing and several runs should be made with the weight in different positions along the circumference of the rotor. A curve representing the variation in amplitude of vibrations, with the angle of location of the weight, can be so obtained. The minimum amplitude will then indicate the true location for the correction weight. In the same manner, by gradual changing the magnitude of the weight, the true magnitude of the correction weight can be established.

In order to simplify the process of determining the location of the correction weight, marking the shaft or recording the vibrations of the shaft end may be very useful. For marking the shaft a special indicator, shown in Fig. 29, is used in the Lawaczeck machine. During vibration the shaft presses against a pencil *ab* suitably arranged and displaces it into a position corresponding to the maximum deflection of the shaft

end so that the end of the marking line on the surface of the shaft determines the angular position at the moment of maximum deflection. Assuming that at resonance the lag of the forced vibrations is equal to $\pi/2$, the location of the disturbing force will be 90° from the point where the marking ceases in the direction of rotation of the shaft. Now the true location for the correction weight can easily be obtained. Due to the fact that near the resonance condition the lag changes very sharply with the speed and also depends on the damping (see Fig. 20, p. 30) two tests are usually necessary for an accurate determination of the location of unbalance. By running the shaft alternately in opposite directions and marking the shaft as explained above the bisector between the two marks determines the axial plane in which the correction weights must be placed.

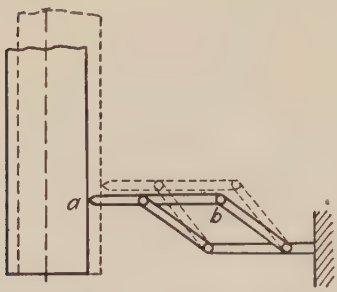


Fig. 29.

For obtaining the location of unbalance by recording the vibrations of the face of the shaft end of the rotor, a special vibration recorder is used in the Lawaczeck machine. The recording paper is attached to the face of the shaft and revolves with the rotor. The pencil of the indicator performs displacements which are the magnified lateral displacements of the shaft end with respect to the immovable pedestal of the machine. In this manner a kind of a polar diagram of lateral vibrations of the shaft will be obtained on the rotating paper attached to the shaft end. By running the shaft twice, in two opposite directions, two diagrams on the rotating paper will be obtained. The axis of symmetry for these two diagrams determines the plane in which the correction weight must be placed.*

The procedure for balancing a rotor AB (see Fig. 28) will now be described. Assume first that the bearing B is locked and the end A of the rotor is free to vibrate in a horizontal axial plane. It has been shown (see paragraph 10, p. 38) that in the most general case the unbalance can be represented by two centrifugal forces acting in two arbitrary

* A more detailed description of methods of balancing by using the Lawaczeck-Heymann machine can be found in the paper by Ernst Lehre: "Der heutige Stand der Auswuchttechnik," Maschinenbau, Vol. 16 (1922–1923), p. 62. See also the paper by E. v. Brauchisch, "Zur Theorie und experimentellen Prufung des Auswuchtens," Zeitschr. f. Angw. Math. und Mech., Vol. 3 (1923), p. 61.

chosen planes perpendicular to the axis of the shaft. Let the force P in the plane I (see Fig. 28, a) and the force Q in the plane II through the center of the locked bearing B represent the unbalance in the rotor. In the case under consideration the force P only will produce vibrations. Proceeding as described above the force P can be determined and the vibrations can be annihilated by a suitable choice of correction weights. In order to balance the force Q, the bearing A must be locked and the bearing B made free to vibrate (see Fig. 28, b). Taking the plane III, for placing the correction weight and proceeding as before, the magnitude and the location of this weight can be determined. Let G denote the centrifugal force corresponding to this weight. Then from the equation of statics,

$$G \cdot c = Q \cdot l$$

and

$$Q = \frac{Gc}{l}. \tag{a}$$

It is easy to see that by putting the correction weight in the plane III, we annihilate vibrations produced by Q only under the condition that the bearing A is locked. Otherwise there will be vibrations due to the fact that the force Q and the force G are acting in two different planes II and III. In order to obtain complete balance one correction weight must be placed in each of the two planes I and III, such that the corre-

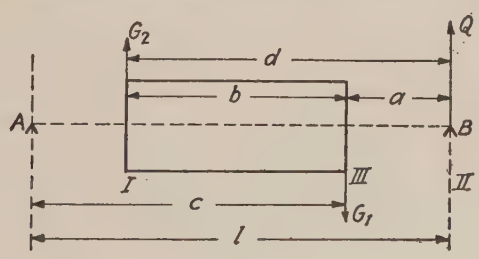


FIG. 30.

sponding centrifugal forces G_1 and G_2 will have as their resultant the force $-Q$ equal and opposite to the force Q (Fig. 30). Then, from statics, we have,

$$G_1 - G_2 = Q,$$
$$G_2 \cdot b = Q \cdot a,$$

from which, by using eq. (a)

$$G_2 = \frac{Qa}{b} = \frac{Gac}{bl}, \tag{b}$$

$$G_1 = Q + G_2 = \frac{Gcd}{bl}. \qquad (c)$$

It is seen from this that by balancing at the end B and determining in this manner the quantity G, the true correction weight for the plane III and the additional correction weight for the plane I can be found from equations (b) and (c) and complete balancing of the rotor will be obtained.

Akimoff's Balancing Machine,* consists of a rigid table on which the rotor and the compensating device are mounted. The table is secured to the pedestals in such a way that it is free to vibrate, either about an axis parallel to the axis of the rotor or about an axis perpendicular to the axis of the rotor. In the first case static unbalance alone produces vibrations; in the second, both static and dynamic unbalance will cause vibration. Beginning with checking for static unbalance, the table must be supported in such a way as to obtain vibration about the axis parallel to the axis of rotation of the rotor. The method for determining the location and magnitude of unbalance consists in creating an artificial unbalance in some moving part of the machine to counteract the unbalance of the body to be tested. When this artificial unbalance becomes the exact counterpart of the unbalance in the body being tested, the whole unit ceases to vibrate and the magnitude and the angular plane of unbalance are indicated on the machine.

After removing the static unbalance of the rotor testing for dynamic unbalance can be made by re-arranging the supports of the table in such a manner as to have the axis of vibration perpendicular to the axis of rotation. The magnitude and the angular plane of dynamic unbalance will then be easily found in the same manner as explained above by introducing an artificial couple of unbalance in the moving part of the machine. It is important to note that all the static unbalance must be removed before checking for dynamic unbalance.†

Balancing in the Field.—Experience with large high speed units shows that while balancing carried out on the balancing machine in the shop may show good results; nevertheless this testing is usually done at comparatively low speed and in service where the operating speeds are high, unbalance may still be apparent due to slight changes in mass

* Trans. A. S. M. E., Vol. 38 (1916).

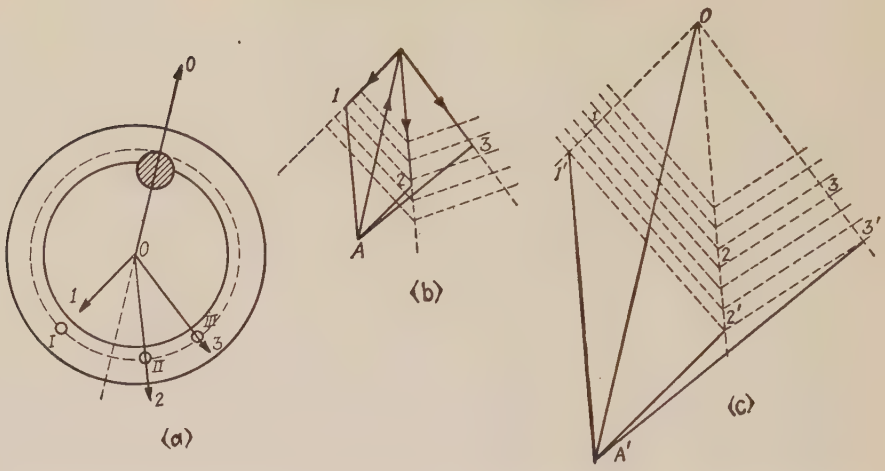
† A balancing machine by which both kinds of unbalance are determined simultaneously has been developed by C. R. Söderberg and W. E. Trumpler. See paper presented by C. R. Söderberg before the A. S. M. E. Spring Meeting (1923).

distribution. It is therefore necessary also to check the balancing condition for normal operating speed. This test is carried out, either in the shop where the rotor is placed for this purpose on rigid bearings or in the field after it is assembled in the machine. The procedure of balancing in such conditions can be about the same as described above in considering the Lawaczeck balancing machine. This consists in consecutive balancing of both ends of the rotor. In correcting the unbalance at one end, it is assumed that vibrations of the corresponding pedestal are produced only by the unbalance at this end.* The magnitude and the location of the correction weight can then be found from measurements of the amplitudes of vibrations of the pedestal, which are recorded by a suitable instrument. Four measurements are necessary for four different conditions of the rotor in order to have sufficient data for a complete solution of the problem. The first measurement must be made for the rotor in its initial condition and the three others for the rotor with some arbitrary weights placed consecutively in three different holes of the balancing ring of the rotor end at which the balancing is being performed. A rough approximation of the location of the correction weight can be found by marking the shaft of the rotor in its initial condition as was explained above (p. 41). The three trial holes must be taken near the found location.

On the basis of these four measurements the determination of the unbalance can now be made on the assumption that the amplitudes of vibration of the pedestal are proportional to the unbalance. Let 00 (Fig. 31, *a*) be the vector representing the unknown original unbalance and let 01, 02, 03 be the vectors corresponding to the trial corrections I, II and III put into the balancing ring of the rotor end during the second, third and fourth runs, respectively. Then vectors $A1$, $A2$, $A3$ (Fig. 31, *b*) represent the resultant unbalances for these three runs. These vectors, according to the assumption made are proportional to the amplitudes of vibration of the pedestal measured during the respective runs.

When balancing a rotor, the magnitudes and directions of 01, 02, 03 (Fig. 31, *b*) are known and a network as shown in Fig. 31, *c* by dotted lines can be constructed. Taking now three lengths $A^1 1^1$, $A^1 2^1$, and $A^1 3^1$ proportional to the amplitudes observed during the trial runs and using the network, a diagram geometrically similar to that given in Fig. 31, *b*, can be constructed (Fig. 31, *c*). The direction $0A^1$ gives then the location of the true correction and the length $0A^1$ represents the weight of it to the same scale as 01^1, 02^1 and 03^1 represent the trial weights I, II and III,

* This assumption is accurate enough in cases of rotors of considerable length.


(a)
(b)
(c)
A
A'
I
II
III
1
2
3
1'
2'
3'

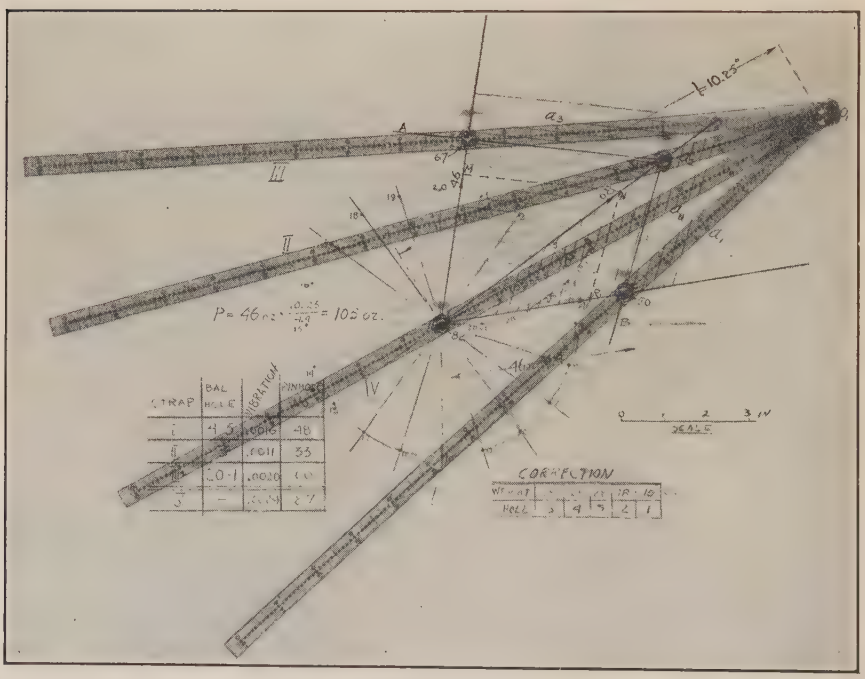

10.25"
P = 46 oz
= 105 oz.
STRAP
BAL HOLE
VIBRATION
PINHOLE
CORRECTION
SCALE

FIG. 31.

respectively. It should be noted that the length $0A^1$ if measured to the same scale as the amplitudes A^11^1, A^12^1, A^13^1, must give the amplitude of the initial vibration of the pedestal, this being a check of the solution obtained. In the photograph 31, *d* a simple device for the solution of this problem, consisting of four straps connected together by a hinge 0_1, is shown.* Taking the same correction weight in three trial runs, the network of Fig. 31, *c* will be replaced by concentric circles such as shown in the photograph. Taking now three lengths a_1, a_2 and a_3 on the straps proportional to the amplitudes observed during the trial runs and moving the ends of these straps along the radii of the correction weights it will make no difficulty to find a position of the system where all these three ends will be situated on the same circle. The corresponding vector 00_1 will then determine the position and the magnitude of the true correction weight.

12. General Case of Disturbing Force.—In the previous discussion of forced vibrations (see paragraph 8) a particular case of a disturbing force proportional to *sin mt* was considered. In general a periodical disturbing force $f(t)$ can be obtained by superimposing simple sine and cosine terms which can be represented in the form of a trigonometrical series such as

$$f(t) = a_0 + a_1 \cos mt + a_2 \cos 2mt + \cdots b_1 \sin mt + b_2 \sin 2mt + \cdots, \quad (a)$$

in which

$$\frac{m}{2\pi} = \text{the frequency of the disturbing force,}$$

$$\tau_1 = \frac{2\pi}{m} = \text{the period of the disturbing force.}$$

In order to calculate any one of the coefficients of eq. (*a*) provided $f(t)$ be known the following procedure must be followed. Assume that any coefficient a_i is desired, then both sides of the equation must be multiplied by $\cos i\, mt\, dt$ and integrated from $t = 0$ to $t = \tau_1$. It can be shown that

$$\int_0^{\tau_1} a_0 \cos i\, mt\, dt = 0; \qquad \int_0^{\tau_1} a_k \cos k\, mt \cos i\, mt\, dt = 0;$$

$$\int_0^{\tau_1} b_k \sin k\, mt \cos i\, mt\, dt = 0; \qquad \int_0^{\tau_1} a_i \cos^2 i\, mt\, dt = \frac{a_i}{2}\tau_1,$$

* This device has been developed by G. B. Karelitz and proved very useful in field balancing. See "Power" Feb. 7 and 14, 1928.

where i and k denote integer numbers 1, 2, 3, $\cdots$. By using these formulas we obtain

$$a_i = \frac{2}{\tau_1}\int_0^{\tau_1} f(t) \cos i\, mt\, dt. \tag{b}$$

In the same manner, by multiplying eq. (a) by $\sin i\, mt\, dt$, we obtain

$$b_i = \frac{2}{\tau_1}\int_0^{\tau_1} f(t) \sin i\, mt\, dt. \tag{c}$$

Finally, multiplying eq. (a) by dt and integrating from $t = 0$ to $t = \tau_1$, we have

$$a_0 = \frac{1}{\tau_1}\int_0^{\tau_1} f(t)dt. \tag{d}$$

It is seen that by using the formulas (b), (c) and (d), the coefficients of eq. (a) can be calculated if $f(t)$ be known analytically. If $f(t)$ be given graphically, while no analytical expression is available, some approximate numerical method for calculating the integrals (b), (c) and (d) must be used or they can be obtained mechanically by using one of the known instruments for analyzing curves in a trigonometrical series.*

Assuming that the disturbing force is represented in the form of a trigonometrical series, the equation for forced vibrations will be (see eq. (32), p. 26)

$$\ddot{x} + 2n\dot{x} + p^2x = a_0 + a_1 \cos mt + a_2 \cos 2mt + \cdots + b_1 \sin mt + b_2 \sin 2mt + \cdots. \tag{e}$$

The general solution of this equation will consist of two parts, one of free vibrations (see eq. (25), p. 23) and one of forced vibrations. The free vibrations will be gradually damped due to friction. In considering the forced vibration it must be noted that in the case of a linear equation, such as eq. (e), the forced vibrations will be obtained by superimposing the forced vibrations produced by every term of the series (a). These latter vibrations can be found in the same manner as explained in paragraph (8) and on the basis of solution (33) (see p. 26) it can be concluded that large forced vibrations may occur when the period of one of the terms of series (a) coincides with the period τ of the natural vibrations of the system, i.e., if the period τ_1 of the disturbing force is equal to or a multiple of the period τ.

* A discussion of various methods of analyzing curves in a trigonometrical series and a description of the instruments for harmonical analysis can be found in the book: "Practical Analysis," by H. von Sanden.

As an example consider the vibrations produced in the frame $ABCD$ by the inertia forces of a horizontal engine (Fig. 32) rotating with constant angular velocity ω. Assume that the horizontal beam BC is very

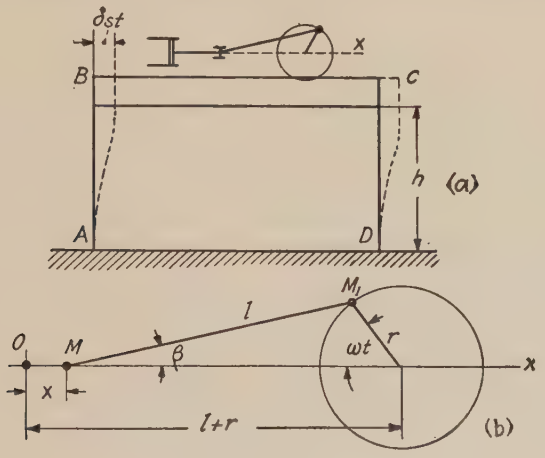


FIG. 32.

rigid and that due to bending of the columns horizontal vibrations alone can be considered. The natural period of these vibrations can easily be obtained. It is only necessary to calculate the statical deflection δ_{st} of the top of the frame under the action of a horizontal force Q equal to the weight of the engine together with the weight of horizontal platform BC. (The mass of the vertical columns is neglected in this calculation). Assuming that the beam BC is absolutely rigid and rests on two columns, we have

$$\delta_{st} = \frac{Q}{3EI}\left(\frac{h}{2}\right)^3.$$

Substituting this in the equation,

$$\tau = 2\pi\sqrt{\frac{\delta_{st}}{g}},$$

the period of natural vibration will be found.

In the case under consideration, forced vibrations will be produced by the inertia forces of the rotating and reciprocating masses of the engine. In considering these forces the mass of the connecting rod can be replaced with sufficient accuracy by two masses, one at the crank pin and the second at the cross-head. To the same two points all other unbalanced

masses in motion easily can be reduced, so that finally only two masses M_1 and M have to be taken into consideration (Fig. 32, *b*). The horizontal component of the inertia force of the mass M_1 is

$$- M_1\omega^2 r \cos \omega t, \tag{f}$$

in which ω = angular velocity of the engine,
r = the radius of the crank,
ωt = the angle of the crank to the x axis.

The motion of the reciprocating masses M is more complicated. Let x denote the displacement of M from the dead position and β, the angle between the connecting rod and the x axis. From the figure we have,

$$x = l(1 - \cos \beta) + r(1 - \cos \omega t) \tag{g}$$

and

$$r \sin \omega t = l \sin \beta. \tag{h}$$

From (*h*),

$$\sin \beta = \frac{r}{l} \sin \omega t.$$

The length l is usually several times larger than r and with a sufficient accuracy it can be assumed that

$$\cos \beta = \sqrt{1 - \frac{r^2}{l^2}\sin^2 \omega t} \approx 1 - \frac{r^2}{2l^2}\sin^2 \omega t.$$

Substituting in eq. (*g*),

$$x = r(1 - \cos \omega t) + \frac{r^2}{2l}\sin^2 \omega t. \tag{k}$$

From this equation the velocity of the reciprocating masses will be

$$\dot{x} = r\omega \sin \omega t + \frac{r^2\omega}{2l}\sin 2\omega t$$

and the corresponding inertia forces will be

$$- M\ddot{x} = - M\omega^2 r \left(\cos \omega t + \frac{r}{l} \cos 2\omega t\right). \tag{l}$$

Combining this with (*f*) the complete disturbing force will be obtained. It will be noted that this expression consists of two terms, one having a frequency equal to the number of revolutions of the machine and another having twice as high a frequency. From this it can be concluded that in the case under consideration we have two critical speeds of the engine:

the first when the number of revolutions of the machine per second is equal to the frequency $1/\tau$ of the natural vibrations of the system and the second when the number of revolutions of the machine is half of the above value. By a suitable choice of the rigidity of the columns AB and CD it is always possible to ascertain conditions sufficiently far away from such critical speeds and to remove in this manner the possibility of large vibrations. It must be noted that the expression (1) for the inertia force of the reciprocating masses was obtained by making several approximations. A more accurate solution will also contain harmonics of a higher order. This means that there will be critical speeds of an order lower than those considered above, but usually these are of no practical importance because the corresponding forces are too small to produce substantial vibrations of the system.

In the above consideration the transient condition was excluded. It was assumed that the free vibrations of the system, usually generated at the beginning of the motion, have been annihilated by damping and forced vibrations alone are being considered. When the displacement of a system at the beginning of the motion is to be investigated the method described in paragraph 5 can be used. It has been shown (see eq. (29), p. 24) that when the damping is proportional to the velocity, the displacement at any moment t, produced by an initial velocity $\dot{x}_0$, is equal to

$$\frac{\dot{x}_0}{p_1} e^{-nt} \sin p_1 t.$$

If $f(t)$ denotes the disturbing force per unit mass of the system the elemental displacement dx at the moment t due to the action of the force $f(t)$ during the interval from t_1 to $t_1 + dt_1$ will be represented by the following expression (see eq. (c), p. 20)

$$dx = \frac{f(t_1)}{p_1} e^{-n(t-t_1)} \sin p_1(t - t_1) dt_1.$$

The complete displacement of the system at the time t, due to the action of the force $f(t_1)$ from $t_1 = 0$ to $t_1 = t$, will then be represented by the following integral

$$x = \frac{1}{p_1} \int_0^t f(t_1) e^{-n(t-t_1)} \sin p_1(t - t_1) dt_1 \cdots. \tag{41}$$

This expression for the complete displacement of the system is especially useful for those cases where $f(t_1)$ cannot be represented with sufficient accuracy by a few terms only of the trigonometrical series (a), as, for

instance, in the case of impact or sharp variations in the impressed force.

13. The Energy Method of Studying Vibration Problems.—The vibration problem of a system such as shown in Fig. 1 (see p. 1) can be easily solved by a consideration of the energy of the system. This energy method is also very useful for an approximate calculation of the frequency of more complicated systems. Considering the arrangement of Fig. 1 and neglecting first the mass of the spring, the kinetic energy of the system during the vibration will be

$$\frac{W}{2g}(\dot{x})^2. \tag{a}$$

The potential energy of the system consists of two parts: first, the potential energy of deformation of the spring and second, the potential energy of the load by virtue of its position. Considering the energy of deformation, the tensile force in the spring corresponding to any displacement x will be $(\delta_{st} + x)k$, and the corresponding energy of the spring will be

$$k\frac{(\delta_{st} + x)^2}{2}.$$

The energy of the spring in the equilibrium position ($x = 0$) is

$$\frac{k\delta_{st}^2}{2},$$

therefore the energy stored in the spring during the displacement x will be

$$k\frac{(\delta_{st} + x)^2}{2} - \frac{k\delta_{st}^2}{2} = k\delta_{st}x + \frac{kx^2}{2} = Wx + \frac{kx^2}{2}. \tag{b}$$

As regards the energy due to position of load we see that this is diminished during the displacement x by the quantity

$$Wx. \tag{c}$$

On the basis of (b) and (c) the complete variation of the potential energy during the displacement x will be equal to

$$\frac{kx^2}{2}. \tag{d}$$

Due to the fact that the weight W is always in balance with the initial tensile force in the spring produced by the statical extension δ_{st}, the final

expression (d) for the potential energy of the system is the same as for the case when $W = 0$, and the elongation of the spring is equal to x. Now, on the basis of the principle of conservation of energy, neglecting the damping, the sum of the kinetic and potential energies of the system remains constant. Therefore, from eqs. (a) and (d), we have,

$$\frac{W}{2g}\dot{x}^2 + \frac{kx^2}{2} = \text{const.} \tag{e}$$

The magnitude of the constant on the right side of the equation (e) depends on the initial conditions. Take, for instance, that at $t = 0$ the displacement is equal to x_0 and the initial velocity is zero, then eq. (e) becomes

$$\frac{W}{2g}\dot{x}^2 + \frac{kx^2}{2} = \frac{kx_0^2}{2}. \tag{f}$$

This means that during vibration the sum of the kinetic and potential energy remains equal to the initial energy of deformation. When during the vibration x becomes equal to x_0 the velocity of the weight W becomes equal to zero and the energy of the system consists of the potential energy only. When x becomes equal to zero, which occurs when the load during vibration passes through its middle position, the velocity has its maximum value and the corresponding magnitude of the kinetic energy, from (f), becomes

$$\frac{W(\dot{x}^2)_{\max}}{2g} = \frac{kx_0^2}{2}. \tag{g}$$

This means that at this moment the total energy of the system is kinetic and equal to the potential energy which was stored in the system during the initial displacement x_0 from the position of equilibrium.

Eq. (g) can be used for calculating the frequency of vibration of the system. It has been previously shown that in the case under consideration we have a simple harmonic motion, i.e., we can take

$$x = x_0 \cos pt \qquad \text{and} \qquad (\dot{x})_{\max} = x_0 p.$$

Substituting this in eq. (g) we obtain

$$p^2 = \frac{kg}{W}.$$

This coincides with eq. (3) (see p. 2) as previously obtained.

The described energy method can be used also for more complicated cases. Consider as an example the vibrations of the *amplitude meter* shown

schematically in Fig. 33. The weight W on the spring k_1 has a low natural frequency and if the amplitude meter is attached to a body vibrating at a high frequency in a vertical direction the weight W remains practically immovable in space and the pointer A, connected with W, indicates on the scale the magnified amplitude of the vibration. In order to obtain the frequency of the free vibrations of the instrument with greater accuracy, not only the weight W and the spring k_1, but also the arm AOB and the spring k_2 must be taken into consideration. Let x denote a small vertical displacement of the weight W from the position of equilibrium. Then the potential energy of the two springs with the scale constants k_1 and k_2 will be

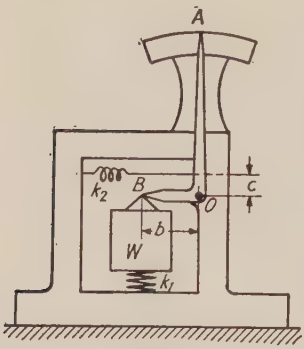


FIG. 33.

$$\frac{k_1 x^2}{2} + \frac{k_2}{2}\left(\frac{c}{b}\right)^2 x^2. \tag{h}$$

The kinetic energy of the weight W will be, as before,

$$\frac{W}{2g}\dot{x}^2. \tag{k}$$

The angular velocity of the arm AOB rotating about the point O is

$$\frac{\dot{x}}{b}$$

and the kinetic energy of the same arm will be

$$\frac{I}{2}\frac{\dot{x}^2}{b^2}. \tag{l}$$

Now the equation of motion, corresponding to the equation (e) above, will be from (h), (k) and (l),

$$\left(\frac{W}{2g} + \frac{I}{2b^2}\right)\dot{x}^2 + \left(\frac{k_1}{2} + \frac{k_2}{2}\frac{c^2}{b^2}\right)x^2 = \text{const.}$$

We see that this equation has the same form as the equation (e); only instead of the mass W/g we have now the *reduced* mass

$$\frac{W}{g} + \frac{I}{b^2}$$

and instead of the spring constant k we have the *reduced spring constant* $k_1 + k_2(c^2/b^2)$.

14. Rayleigh Method.—In all the previously considered cases, such as shown in Figs. 1, 3, and 4, by using certain simplifications the problem was reduced to the simplest case of vibration of a system with one degree of freedom. For instance, in the arrangement shown in Fig. 1, the mass of the spring was neglected in comparison with the mass of the weight W, while in the arrangement shown in Fig. 3 the mass of the beam was neglected and again in the case shown in Fig. 4 the moment of inertia of the shaft was neglected in comparison with the moment of inertia of the disc. Although these simplifications are accurate enough in many practical cases, there are several technical problems in which a more detailed consideration of the accuracy of such approximations becomes necessary.

In order to determine the effect of such simplifications on the frequency of vibration an approximate method developed by Lord Rayleigh * will now be discussed. In applying this method some assumption regarding the configuration of the system during vibration has to be made. The frequency of vibration will then be found from a consideration of the energy of the system. As a simple example of the application of Rayleigh's method we take the case shown in Fig. 1 and discussed in paragraph 13.

Assuming that the mass of the spring is small in comparison with the mass of the load W, the *type of vibration* will not be substantially affected by the mass of the spring and with a sufficient accuracy it can be assumed that the displacement of any cross section of the spring at a distance c from the fixed end is the same as in the case of a massless spring, i.e., equal to

$$\frac{xc}{l}, \tag{a}$$

where l = the length of the spring.

If the displacements, as assumed above, are not affected by the mass of the spring, the expression for the potential energy of the system will be the same as in the case of a massless spring, (see eq. (d), p. 52) and only the kinetic energy of the system has to be reconsidered. Let q denote the weight of the spring per unit length. Then the mass of an element of the spring of length dc will be qdc/g and the corresponding kinetic

* See Lord Rayleigh's book "Theory of Sound," 2d Ed., Vol. 1, pp. 111 and 287.

energy, by using eq. (a), becomes

$$\frac{q}{2g}\left(\frac{\dot{x}c}{l}\right)^2 dc.$$

The complete kinetic energy of the spring will be

$$\frac{q}{2g}\int_0^l \left(\frac{\dot{x}c}{l}\right)^2 dc = \frac{\dot{x}^2}{2g}\frac{ql}{3}.$$

This must be added to the kinetic energy of the weight W; so that the equation of conservation of energy becomes

$$\frac{\dot{x}^2}{2g}\left(W + \frac{ql}{3}\right) + \frac{kx^2}{2} = \frac{kx_0^2}{2}. \tag{b}$$

Comparing this with eq. (f) of the previous paragraph it can be concluded that in order to estimate the effect of the mass of the spring on the period of natural vibration it is only necessary to add one-third of the weight of the spring to the weight W.

This conclusion, obtained on the assumption that the weight of the spring is very small in comparison with that of the load, can be used with sufficient accuracy even in cases where the weight of the spring is of the same order as W. For instance, for $ql = .5W$, the error of the approximate solution is about ½%. For $ql = W$, the error is about ¾%. For $ql = 2W$, the error is about 3%.*

As a second example consider the case of vibration of a beam of uniform cross section loaded at the middle (see Fig. 34). If the weight ql of the beam is small in comparison with the load W, it can be assumed with sufficient accuracy that the deflection curve of the beam during vibration has the same shape as the statical deflection curve. Then, denoting by x the displacement of the load W during vibration, the displacement of any element qdc of the beam, distant c from the support, will be,

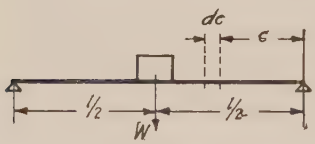


Fig. 34.

$$x \cdot \frac{3cl^2 - 4c^3}{l^3}.$$

The kinetic energy of the beam itself will be,

$$2\int_0^{l/2} \frac{q}{2g}\left(\dot{x}\frac{3cl^2 - 4c^3}{l^3}\right)^2 dc = \frac{17}{35}ql\frac{\dot{x}^2}{2g}. \tag{42}$$

* A more detailed consideration of this problem is given in paragraph 48.

This kinetic energy of the vibrating beam must be added to the energy $W\dot{x}^2/2g$ of the load concentrated at the middle in order to estimate the effect of the weight of the beam on the period of vibration, i.e., the period of vibration will be the same as for a massless beam loaded at the middle by the load

$$W + 17/35ql.$$

It must be noted that eq. (42) obtained on the assumption that the weight of the beam is small in comparison with that of the load W, can be used in all practical cases. Even in the extreme case where $W = 0$ and where the assumption is made that $17/35ql$ is concentrated at the middle of the beam, the accuracy of the approximate method is sufficiently good. The deflection of the beam under the action of the load $17/35ql$ applied at the middle is,

$$\delta_{st} = \frac{17}{35} ql \cdot \frac{l^3}{48EI}.$$

Substituting this in eq. (5) (see p. 2) the period of the natural vibration will be

$$\tau = 2\pi \sqrt{\frac{\delta_{st}}{g}} = .632 \sqrt{\frac{ql^4}{EIg}}.$$

The exact solution for this case * is

$$\tau = \frac{2}{\pi} \sqrt{\frac{ql^4}{EIg}} = .637 \sqrt{\frac{ql^4}{EIg}}.$$

It is seen that the error of the approximate solution for this limiting case is less than 1%.

The same method can be applied also in the case shown in Fig. 35. Assuming that during the vibration the shape of the deflection curve of the beam is the same as the one produced by a load statically applied at the end and denoting by x the vertical displacement of the load W the kinetic energy of the cantilever beam of uniform cross section will be,

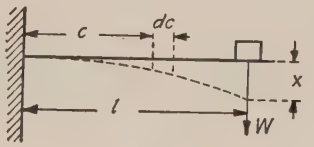


FIG. 35.

$$\int_0^l \frac{q}{2g} \left(\dot{x} \frac{3c^2l - c^3}{2l^3} \right)^2 dc = \frac{33}{140} ql \frac{\dot{x}^2}{2g}. \tag{43}$$

The period of vibration will be the same as for a massless cantilever beam

* See paragraph 42, eq. 137[1].

loaded at the end by the load,

$$W + 33/140ql.$$

This result was obtained on the assumption that the weight ql of the beam is small in comparison with W, but it is also accurate enough for cases where ql is not small. Applying the result to the extreme case where $W = 0$ we obtain

$$\delta_{st} = \frac{33}{140} ql \frac{l^3}{3EI}.$$

The corresponding period of vibration will be

$$\tau = 2\pi \sqrt{\frac{\delta_{st}}{g}} = \frac{2\pi}{3.567} \sqrt{\frac{ql^4}{EIg}}. \tag{c}$$

The exact solution for the same case is *

$$\tau = \frac{2\pi}{3.515} \sqrt{\frac{ql^4}{EIg}}. \tag{d}$$

It is seen that the error of the approximate solution is about 1½%. For the case $W = 0$ a better approximation can be obtained. It is only necessary to assume that during the vibration the shape of the deflection curve of the beam is the same as the one produced by a uniformly distributed load. The deflection y_0 at any cross section distant c from the built-in section will then be given by the following equation,

$$y_0 = x_0 \left\{ -1/3 + 4/3 \frac{c}{l} + 1/3 \left(1 - \frac{c}{l}\right)^4 \right\}, \tag{e}$$

in which

$$x_0 = \frac{ql^4}{8EI}$$

represents the deflection of the end of the cantilever.

The potential energy of bending will be

$$V = \frac{q}{2} \int_0^l y_0 dc = \frac{8}{5} \cdot \frac{EIx_0^2}{l^3}.$$

The kinetic energy of the vibrating beam is

$$T = \frac{1}{2} \int_0^l \frac{q}{g} \dot{y}^2 dc.$$

Taking (see p. 53)

$$y = y_0 \cos pt \qquad \text{and} \qquad (\dot{y})_{\max} = y_0 p.$$

* See paragraph 43.

The equation for determining p will be (see eq. (g), p. 53)

$$\frac{1}{2}\int_0^l \frac{q}{g}(y_0 p)^2 dc = \frac{8}{5}\frac{EIx_0^2}{l^3}.$$

Substituting (e) for y_0 and performing the integration, we obtain

$$p = 3.530\sqrt{\frac{EIg}{ql^4}}.$$

The corresponding period of vibration is

$$\tau = \frac{2\pi}{3.530}\sqrt{\frac{ql^4}{EIg}}. \qquad (f)$$

Comparing this result with the exact solution (d) it can be concluded that in this case the error of the approximate solution is only about ½%.

It must be noted that an elastic beam represents a system with an infinitely large number of degrees of freedom. It can, like a string, perform vibrations of various types. The choosing of a definite shape for the deflection curve in using Rayleigh's method is equivalent to introducing some additional constraints which reduce the system to one having one degree of freedom. Such additional constraints can only increase the rigidity of the system, i.e., increase the frequency of vibration. In all cases considered above the approximate values of the frequencies as obtained by Rayleigh's method are somewhat higher than their exact values.

In the case of torsional vibrations (see Fig. 4) the same approximate method can be used in order to calculate the effect of the inertia of the shaft on the frequency of the torsional vibrations. Let i denote the moment of inertia of the shaft per unit length. Then assuming that the type of vibration is the same as in the case of a massless shaft the angle of rotation of a cross section at a distance c from the fixed end of the shaft is $c\varphi/l$ and the kinetic energy of one element of the shaft will be

$$\frac{idc}{2}\left(\frac{c\dot{\varphi}}{l}\right)^2.$$

The kinetic energy of the complete shaft will be

$$\frac{i}{2}\int_0^l \left(\frac{c\dot{\varphi}}{l}\right)^2 dc = \frac{\dot{\varphi}^2}{2}\frac{il}{3}. \qquad (44)$$

This kinetic energy must be added to the kinetic energy of the disc in order to estimate the effect of the mass of the shaft on the frequency of

vibration, i.e., the period of vibration will be the same as for a massless shaft having at the end a disc, the moment of inertia of which is equal to

$$I + il/3.$$

The application of Rayleigh's method for calculating the critical speed of a rotating shaft will be shown in the following paragraph.

15. Critical Speed of a Rotating Shaft.—It is well known that rotating shafts at certain speeds become dynamically unstable and large vibrations are likely to develop. This phenomenon is due to resonance effects and a simple example will show that the *critical speed* for a shaft is that speed at which the number of revolutions per second of the shaft is equal to the frequency of its natural lateral vibration.*

Shaft with One Disc.—In order to exclude from our consideration the effect of the weight of the shaft and so make the problem as simple as possible, a vertical shaft with one circular disc will be taken (Fig. 36, *a*). Let C be the center of gravity of the disc and e a small eccentricity, i.e., the distance of C from the axis of the shaft. During rotation, due to the eccentricity e, a centrifugal force will act on the shaft and will produce deflection. The magnitude of the deflection x can easily be obtained from the condition of equilibrium of the centrifugal force and the reactive force P of the deflected shaft. This latter force is proportional to the deflection x, and can be represented in the following form,

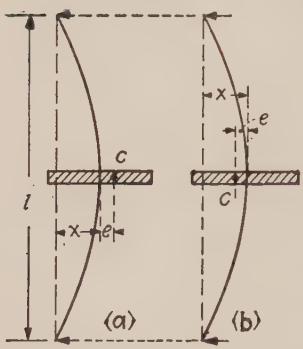


Fig. 36.

$$P = kx.$$

The magnitude of the factor k can be calculated provided the dimensions of the shaft and the conditions at the supports be known. Assuming, for instance, that the shaft has a uniform section and the disc is in the middle between the supports, we have

$$k = \frac{48EI}{l^3}.$$

Now for the condition of equilibrium the following equation for deter-

* A more detailed discussion of lateral vibrations of a shaft is given in paragraph 32.

mining x will be obtained

$$\frac{W}{g}(x + e)\omega^2 = kx, \tag{a}$$

in which

$$\frac{W}{g} = \text{the mass of the disc,}$$

$$\omega = \text{angular velocity of the shaft.}$$

From eq. (a) we have,

$$x = \frac{e}{\frac{k}{\omega^2}\frac{g}{W} - 1}. \tag{b}$$

Remembering (see eq. (3), p. 2) that

$$\frac{kg}{W} = p^2,$$

it can be concluded from (b) that the deflection x tends to increase rapidly as ω approaches p, i.e., when the number of revolutions per second of the shaft approaches the frequency of the lateral vibrations of the shaft and disc. The critical value of the speed will be

$$\omega_{cr} = \sqrt{\frac{kg}{W}}. \tag{44}$$

At this speed the denominator of (b) becomes zero and large lateral vibrations in the shaft occur. It is interesting to note that at speeds higher than the critical quiet running conditions will again prevail. The experiments show that in this case the center of gravity C will be situated between the line joining the supports and the deflected axis of the shaft as shown in Fig. 36, b. The equation for determining the deflection will be

$$\frac{W}{g}(x - e)\omega^2 = kx,$$

from which

$$x = \frac{e}{1 - \frac{kg}{\omega^2 W}}. \tag{c}$$

It is seen that now with increasing ω the deflection x decreases and approaches the limit e, i.e., at very high speeds the center of gravity

of the disc approaches the line joining the supports and the deflected shaft rotates about the center of gravity C.

Shaft Loaded with Several Discs.—It has been shown above in a simple example that the critical number of revolutions per second of a shaft is equal to the frequency of the natural lateral vibration of this shaft. Determining this frequency by using Rayleigh's method the critical speed for a shaft with many discs (Fig. 37) can easily be established. Let W_1, W_2, W_3 denote the loads and x_1, x_2, x_3 denote the corresponding deflections. Then the potential energy of deformation stored in the beam during bending will be

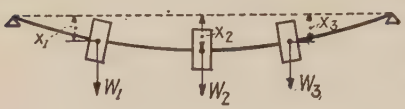


FIG. 37.

$$V = \frac{W_1x_1}{2} + \frac{W_2x_2}{2} + \frac{W_3x_3}{2}. \qquad (d)$$

In calculating the period of the slowest type of vibration the static deflection curve shown in Fig. 37 can be taken as a good approximation for the deflection curve of the beam during vibration. The vertical displacements of the loads W_1, W_2 and W_3 during vibration can be written as:

$$x_1 \cos pt, \qquad x_2 \cos pt, \qquad x_3 \cos pt. \qquad (e)$$

Then the maximum deflections of the shaft from the position of equilibrium are the same as those given in Fig. 37; therefore, the increase in the potential energy of the vibrating shaft during its deflection from the position of equilibrium to the extreme position will be given by equation (d). On the other hand the kinetic energy of the system becomes maximum at the moment when the beam, during vibration, passes through its middle position. It will be noted from eq. (e) that the velocities of the loads corresponding to this position are:

$$px_1, \quad px_2, \quad px_3$$

and the kinetic energy of the system becomes

$$\frac{p^2}{2g}(W_1x_1^2 + W_2x_2^2 + W_3x_3^3). \qquad (f)$$

Equating (d) and (f), the following expression for p^2 will be obtained:

$$p^2 = \frac{g(W_1x_1 + W_2x_2 + W_3x_3)}{W_1x_1^2 + W_2x_2^2 + W_3x_3^2}. \qquad (45)$$

The period of vibration is

$$\tau = \frac{2\pi}{p} = 2\pi\sqrt{\frac{W_1x_1^2 + W_2x_2^2 + W_3x_3^2}{g(W_1x_1 + W_2x_2 + W_3x_3)}}. \tag{46}$$

In general, when n loads are acting on the shaft the period of vibration will be

$$\tau = 2\pi\sqrt{\frac{\sum_1^n W_ix_i^2}{g\sum_1^n W_ix_i}}. \tag{47}$$

It is seen that for calculating τ the statical deflections x_1, x_2, $\cdots$ of the shaft alone are necessary. These quantities can easily be obtained by the usual methods. If the shaft has a variable cross section a graphical method for obtaining the deflections has to be used. The effect of the weight of the shaft itself also can be taken into account. It is necessary for this purpose to divide the shaft into several parts, the weights of which, applied at their respective centers of gravity, must be considered as concentrated loads.

Take, for instance, the shaft shown in figure 38, *a*, the diameters of which the loads acting on it are shown in the figure. By constructing the polygon of forces (Fig. 38, *b*) and the corresponding funicular polygon (Fig. 38, *c*) the bending moment diagram will be obtained. In order to calculate the numerical value of the bending moment at any cross section *mn* of the shaft it is only necessary to measure the corresponding ordinate *e* of the moment diagram to the same scale as used for the length of the shaft and multiply it with the pole distance h measured to the scale of forces in the polygon of forces (in our case $h = 80{,}000$ lbs.). In order to obtain the deflection curve a construction of the second funicular polygon is necessary in which construction the bending moment diagram obtained above must be considered as an imaginary loading diagram. In order to take into account the variation in cross section of the shaft, the intensity of this imaginary loading at every section should be multiplied by I_0/I where I_0 = moment of inertia of the largest cross section of the shaft and I = moment of inertia of the portion of the shaft under consideration. In this manner the final imaginary loading represented by the shaded area (Fig. 38, *c*) is obtained. Subdividing this area into several parts, measuring the areas of these parts in square inches and multiplying them with the pole distance h measured in pounds, the imaginary loads measured in pounds-inches^2 will be obtained.

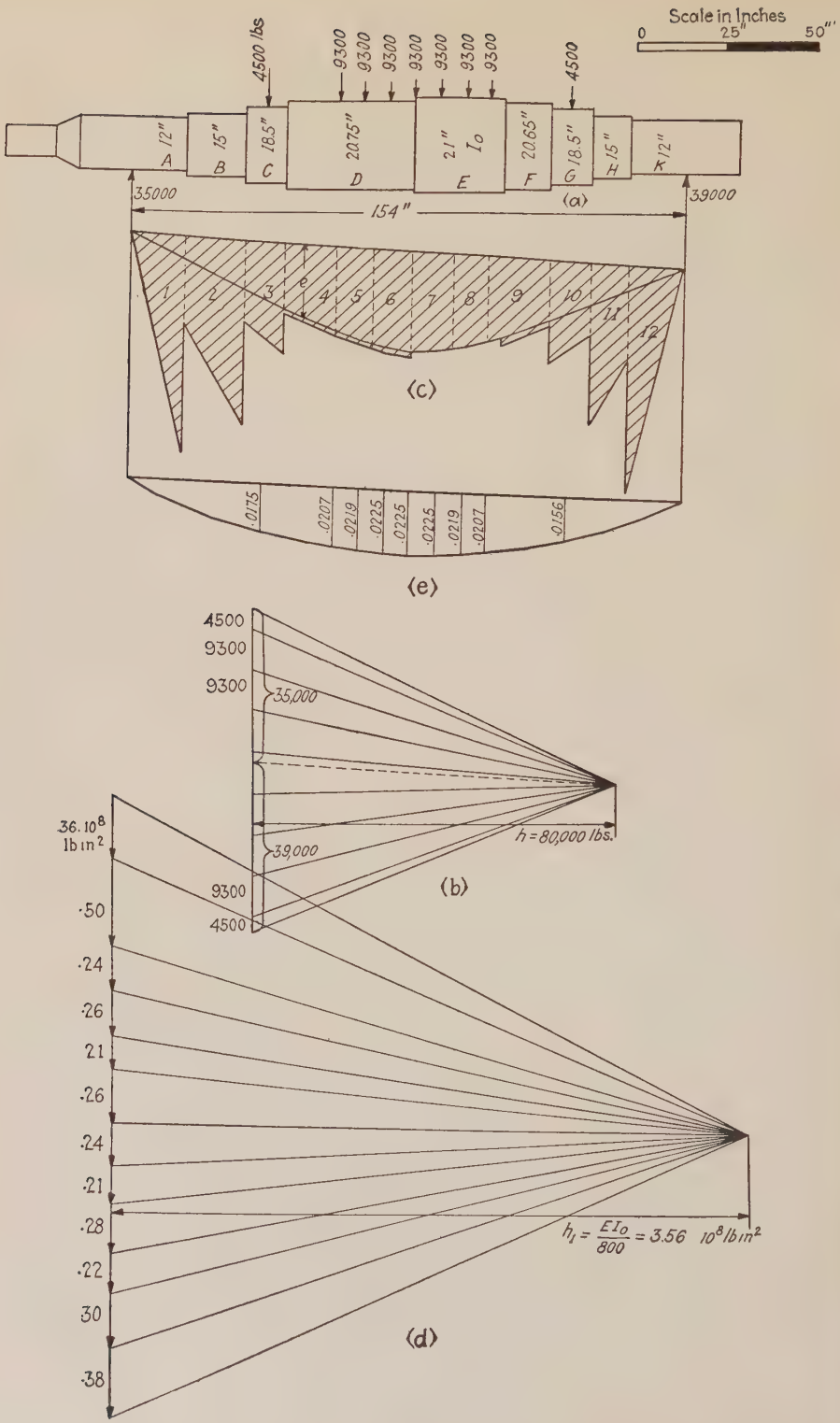

Scale in Inches
0
25"
50"
4500 lbs
9300
9300
9300
9300
9300
9300
9300
4500
12"
A
15"
B
18.5"
C
20.75"
D
21"
I_0
E
20.65"
F
18.5"
G
15"
H
12"
K
35000
39000
154"
(a)
1
2
3
e
4
5
6
7
8
9
10
11
12
(c)
.0175
.0207
.0219
.0225
.0225
.0225
.0219
.0207
.0156
(e)
4500
9300
9300
35,000
39,000
9300
4500
h = 80,000 lbs.
(b)
36.10⁸ lb in²
·50
·24
·26
·21
·26
·24
·21
·28
·22
·30
·38
$h_1 = \frac{EI_0}{800} = 3.56 \ 10^8 \ lb\,in^2$
(d)

FIG. 38.

For these loads, the second polygon of forces (Fig. 38, *d*) is constructed by taking a pole distance h_1 equal to EI_0/n where EI_0 is the largest flexural rigidity of the shaft and n is an integer (in our case $n = 200$). It should be noted that the imaginary loads and the pole distances EI/n have the same dimension, i.e., in.^{2}-lbs., and should be represented in the polygon of forces to the same scale. By using the second polygon of forces the second funicular polygon (Fig. 38, *e*) and the deflection curve of the shaft tangent to this polygon can easily be constructed. In order to get the numerical values of the deflections e it is only necessary to measure them to the same scale to which the length of the shaft is drawn and divide them by the number n used above in the construction of the second polygon. All numerical results obtained from the drawing and necessary in using equation (47) are given in the table below.

W lbs.	$x_i \times 10^2$ in.	$x_i^2 \times 10^4$ in.^{2}	Wx_i lbs. × in.	Wx_i^2 lbs. × in.^{2}
4500	1.75	3.05	79	1.37
9300	2.07	4.28	193	3.98
9300	2.19	4.80	204	4.47
9300	2.25	5.06	209	4.71
9300	2.25	5.06	209	4.71
9300	2.25	5.36	209	4.71
9300	2.19	4.88	204	4.47
9300	2.07	4.28	193	3.98
4500	1.56	2.43	70	1.09
			$\Sigma Wx_i = 1570$	$\Sigma Wx_i^2 = 33.09$

The critical number of revolutions per minute will be obtained now as follows:

$$N_{cr} = \frac{60}{\tau} = \frac{30}{\pi}\sqrt{\frac{g\sum_1^n W_i x_i}{\sum_1^n W_i x_i^2}} = \frac{30}{\pi}\sqrt{\frac{386 \times 1570}{33.09}} = 1290 \text{ R.P.M.}$$

It should be noted that the hubs of spiders or flywheels shrunk on the shaft increase the stiffness of the shaft and may raise its critical speed considerably. In considering this phenomenon it can be assumed that the stresses due to vibration are small and the shrink fit pressure between the hub and the shaft is sufficient to prevent any relative motion between these two parts, so that the hub can be considered as a portion of shaft of an enlarged diameter. Therefore the effect of the hub on the critical

speed will be obtained by introducing this enlarged diameter in the graphical construction developed above.*

In the case of a grooved rotor (Fig. 39) if the distances between the

Fig. 39.

grooves are of the same order as the depth of the groove, the material between two grooves does not take any bending stresses and the flexibility of such a rotor is near to one of the diameter d measured at the bottom of the grooves.†

It must be noted also that in Fig. 37 rigid supports were assumed. In certain cases the rigidity of the supports is small enough so as to produce a substantial effect on the magnitude of the critical speed. If the additional flexibility, due to deformation of the supports, is the same in a vertical and in a horizontal direction the effect of this flexibility can be easily taken into account. It is only necessary to add to the deflections x_1, x_2 and x_3 of the previous calculations the vertical displacement due to the deformation of the supports under the action of the loads W_1, W_2 and W_3. Such additional deflections will lower the critical speed of the shaft.

* Prof. A. Stodola in his book "Dampf- und Gas Turbinen," 6th ed. (1924), p. 383, gives an example where such a consideration of the stiffening effect of shrunk on parts gave a satisfactory result and the calculated critical speed was in good agreement with the experiment. See also paper by B. Eck, Versteifender Einfluss der Turbinenscheiben, V. D. I., Germany, Bd. 72, 1928, S. 51.

† B. Eck, loc. cit.

CHAPTER II

NON-HARMONIC VIBRATIONS

16. General.—All vibration problems contained in the preceding chapter were of such a kind that their consideration involved the solution of the differential equation,

$$\ddot{x} + 2n\dot{x} + p^2x = f(t), \qquad (a)$$

the coefficients of which were considered as constants. It was assumed that the damping and the spring constant of the system did not depend on the displacement x or on the time t. In many technical problems, such assumptions can be made without great error so that the solution of equation (a) for such cases represents with sufficient accuracy the actual motion of the system. On the basis of this solution the following conclusions have been established: (1) The frequency of the free vibration of such systems does not depend on the amplitude, i.e., the vibrations are isochronic. (2) In the case of a periodical disturbing force the vibrations of the system can be divided into two parts, free vibrations and forced vibrations. (3) The lag of a forced harmonic vibration of a given frequency does not depend on the initial conditions of motion and remains constant during the vibration. (4) If forced vibrations are produced by several forces the resultant motion will be obtained by superimposing the vibrations produced by the individual forces. (5) If the frequency of the disturbing force coincides with that of the natural vibration of the system a resonance condition occurs and a large increase in amplitude of the forced vibrations must be expected. These five points are characteristic of simple harmonic motion.

Cases may be encountered, however, in which equation (a) with constant coefficients can be applied only on the assumption that the displacements x are very small. There are also cases in which the condition of the problem can only be described by a differential equation with variable coefficients.

In the following discussion two important groups of vibrations will be considered: (1) vibrations in which the flexibility of the system depends on the displacement x and which will be called in the further discussion *pseudoharmonic vibrations*, and (2) vibrations in which the spring constant depends on the time t and which will be called *quasiharmonic vibrations*.

Exact solutions for both of these problems are known only in certain simple cases and in the consideration of actual technical problems approximate graphical or numerical methods for obtaining a solution are usually applied.

17. Illustrations of Pseudoharmonic Vibration.—There are cases where the spring constant of the vibrating system varies with the displacement, i.e., the restoring force is no longer proportional to the displacement. Such conditions we obtain if the material of the spring does not follow Hooke's law. Sometimes, for instance, an organic material such as rubber or leather is used in couplings and vibration absorbers. The tensile test diagram for these materials has the shape shown in Fig. 40;

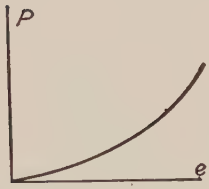


Fig. 40.

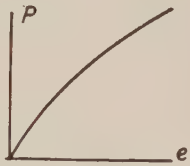


Fig. 41.

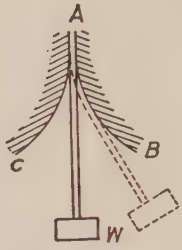


Fig. 42.

the modulus of elasticity increases with the elongation. For small amplitudes of vibration this variation in modulus may be negligible but with increasing amplitude the increase in modulus may result in a substantial increase in the frequency of vibration.

Another example of variable flexibility is met with in the case of structures made of such materials as cast iron or concrete. In both cases the tensile test diagram has the shape, shown in Fig. 41, i.e., the modulus of elasticity decreases with the deformation. Therefore some decrease in the frequency with increase of amplitude of vibration must be expected.

Sometimes special types of steel springs are used, such that their elastic characteristics vary with the displacement. The natural frequency of systems involving such springs depends on the magnitude of amplitude. By using such types of springs the unfavorable effect of resonance can be diminished. If, due to resonance, the amplitude of vibration begins to increase the frequency of the vibration changes, i.e., the resonance condition disappears. A simple example of such a spring is shown in Fig. 42. The flat spring, supporting the weight W, is built in at the end A. During vibration the spring is partially in contact with

one of two cylindrical surfaces AB or AC. Due to this fact the free length of the cantilever varies with the amplitude so that the rigidity of the spring increases with increasing deflection. The conditions are the same as in the case represented in Fig. 40, i.e., the frequency of vibration increases with an increase in amplitude.

If the dimensions of the spring and the shape of the curves AB and AC are known, a curve representing the restoring force as a function of the deflection of the end of the spring can easily be obtained.

As another example of non-harmonic motion the vibration along the x axis of a mass m attached to a stretched wire AB (Fig. 43) will now be considered. Let

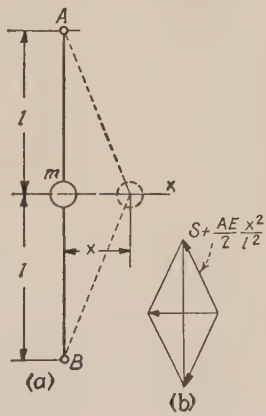


FIG. 43.

S = initial tensile force in the wire,

x = small displacement of the mass m in a horizontal direction,

A = cross sectional area of the wire,

E = modulus of elasticity of the wire.

Then the unit elongation of the wire, due to a displacement x, will be

$$\frac{\sqrt{l^2 + x^2} - l}{l} = \frac{x^2}{2l^2}.$$

The corresponding tensile force in the wire is

$$S + AE \cdot \frac{x^2}{2l^2},$$

and the restoring force acting on the mass m (see Fig. 43, b) will be

$$\left(S + AE \cdot \frac{x^2}{2l^2}\right) \frac{2x}{\sqrt{l^2 + x^2}} = \frac{2Sx}{l} + AE\frac{x^3}{l^3}.$$

The differential equation of motion of the mass m thus becomes

$$m\ddot{x} + \frac{2Sx}{l} + AE\frac{x^3}{l^3} = 0. \qquad (a)$$

It is seen that in the case of very small displacements and when the

initial tensile force S is sufficiently large the last term on the left side of equation (a) can be neglected and a simple harmonic vibration of the mass m in a horizontal direction will be obtained. Otherwise, all three terms of eq. (a) must be taken into consideration. In such a case the restoring force will increase in greater proportion than the displacement and the frequency of vibration will increase with the amplitude.

In the case of a simple mathematical pendulum (Fig. 44) by applying d'Alembert's principle and by projecting the weight W and the inertia force along the direction of the tangent mn the following equation of motion will be obtained:

$$\frac{Wl}{g}\ddot{\theta} + W \sin \theta = 0$$

or

$$\ddot{\theta} + \frac{g}{l} \sin \theta = 0, \qquad (b)$$

in which l = length of the pendulum,

θ = angle between the pendulum and the vertical.

It is seen that only in the case of small amplitudes, when $\sin \theta \approx \theta$, the

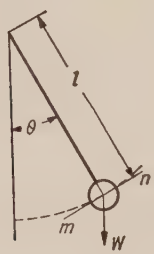


Fig. 44.

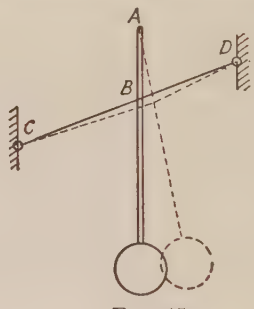


Fig. 45.

oscillations of such a pendulum can be considered as simple harmonic. If the amplitudes are not small a more complicated motion takes place and the period of oscillation will depend on the magnitude of the amplitude. It is clear that the restoring force is not proportional to the displacement but increases at a lesser rate so that the frequency will decrease with an increase in amplitude of vibration. Expanding $\sin \theta$ in a power series and taking only the two first terms of the series, the following equation, instead of eq. (b), will be obtained,

$$\ddot{\theta} + \frac{g}{l} (\theta - 1/6\theta^3) = 0. \qquad (c)$$

Comparing this equation with eq. (a) it is easy to see that by combining the pendulum with a stretched string (Fig. 45) attached to the bar of the pendulum at B and perpendicular to the plane of oscillation, a better approximation to isochronic oscillations may be obtained.

In Fig. 46 another example is given of a system in which the period

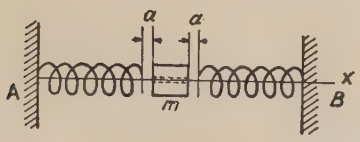


FIG. 46.

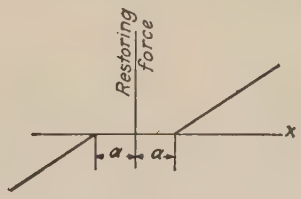


FIG. 47.

of vibration depends on the amplitude. A mass m performs vibrations between two springs by sliding without friction along the bar AB. Measuring the displacements from the middle position of the mass m the variation of the restoring force with the displacement can be represented graphically as shown in Fig. 47. The frequency of the vibrations will depend not only on the spring constant but also on the magnitude of the clearance a and on the initial conditions. Assume, for instance, that at the initial moment ($t = 0$) the mass m is in its middle position and has an initial velocity v in the x direction. Then the time necessary to cross the clearance a will be

$$t_1 = \frac{a}{v}. \tag{d}$$

After crossing the clearance, the mass m comes in contact with the spring and the further motion in the x direction will be simple harmonic. The time during which the velocity of the mass is changing from v to 0 (quarter period of the simple harmonic motion) will be (see eq. (5), p. 2)

$$t_2 = \frac{\pi}{2}\sqrt{\frac{m}{k}}, \tag{e}$$

where k = spring constant. The complete period of vibration of the mass m will be

$$\tau = 4(t_1 + t_2) = \frac{4a}{v} + 2\pi\sqrt{\frac{m}{k}}. \tag{f}$$

For a given magnitude of clearance, a given mass m and a given spring constant k the period of vibration depends only on the initial velocity v.

The period becomes very large for small values of v and decreases with increase of v, approaching the limit $\tau_0 = 2\pi\sqrt{m/k}$ (see Fig. 48) when $v = \infty$. Such conditions always obtain if there are clearances in the system between the vibrating mass and the spring.

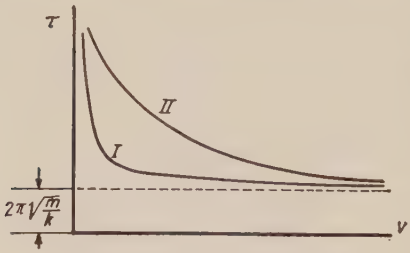


Fig. 48.

If the clearances are very small, the period τ remains practically constant for the larger part of the range of the speed v, as shown in Fig. 46 by curve 1. With increase in clearance for a considerable part of the range of speed v a pronounced variation in period of vibration takes place (see curve 2 in Fig. 48). The period of vibration of such a system may have any value between $\tau = \infty$ and $\tau = \tau_0$. If a periodic disturbing force, having a period larger than τ_0, is acting, it will always be possible to give to the mass m such an impulse that the corresponding period of vibration will become equal to τ and in such manner resonance conditions will be established. Some heavy vibrations in electric locomotives have been explained in this manner.*

18. Free Pseudoharmonic Vibrations.—If damping be neglected the general equation of motion in this case has the form

$$\frac{W}{g}\ddot{x} + k^2 f(x) = 0 \tag{a}$$

or

$$\ddot{x} + p^2 f(x) = 0, \tag{48}$$

in which $p^2 f(x)$ represents the restoring force per unit mass as a function of the displacement x. In order to get the first integral of eq. (48) we multiply it by dx/dt, then it can be represented in the following form:

$$\frac{dx}{dt}\, d\left(\frac{dx}{dt}\right) + p^2 f(x)dx = 0$$

or

$$1/2\, d\left(\frac{dx}{dt}\right)^2 + p^2 f(x)dx = 0,$$

from which, by integration we obtain

$$1/2\left(\frac{dx}{dt}\right)^2 + p^2 \int f(x)dx = 0. \tag{b}$$

* See A. Wichert, Schüttelerscheinungen bei elektrischen Lokomotiven, Forschungsarbeiten, Heft 266 (1924).

If $f(x)$ and the initial conditions are known, the velocity of motion for any position of the system can be calculated from eq. (b). Assume, for instance, that the variation in the restoring force with the displacement is given by curve Om (see Fig. 49) and that in the initial moment $t = 0$, the system has a displacement equal to x_0 and an initial velocity equal to zero. Then, from eq. (b), for any position of the system we have

$$1/2\left(\frac{dx}{dt}\right)^2 = p^2\int_x^{x_0} f(x)dx, \qquad (c)$$

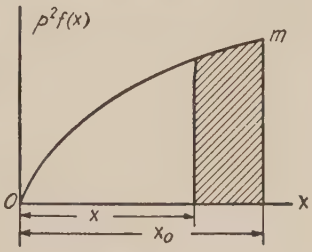


FIG. 49.

which means that at any position of the system the kinetic energy is equal to the difference of the potential energy which was stored in the spring in the initial moment, due to x_0 and the potential energy at the moment under consideration. In Fig. 49 this decrease in potential energy is shown by the shaded area. From eq. (c), we have*

$$dt = \frac{dx}{-\sqrt{2p^2\int_x^{x_0} f(x)dx}}. \qquad (d)$$

By integration of this equation, the time t as a function of the displacement will be obtained,

$$t = \int \frac{dx}{-\sqrt{2p^2\int_x^{x_0} f(x)dx}}. \qquad (e)$$

Take, for instance, as an example, the case of simple harmonic vibration. Then

$$f(x) = x.$$

From eq. (e), we obtain

$$t = \int \frac{dx}{-p\sqrt{x_0^2 - x^2}} = \int \frac{d\left(\frac{x}{x_0}\right)}{-p\sqrt{1-\left(\frac{x}{x_0}\right)^2}}$$

or

$$t = \frac{1}{p}\text{arc cos}\frac{x}{x_0},$$

from which,

$$x = x_0 \cos pt.$$

* The minus sign is taken because in our case an increase in time is connected with a decrease of x.

This result coincides with what we had before for simple harmonic motion.

As a second example, assume,

$$f(x) = x^{2n-1}.$$

Substituting this in eq. (e), we obtain

$$t = \frac{\sqrt{n}}{p} \int \frac{dx}{-\sqrt{x_0^{2n} - x^{2n}}}.$$

The period of vibration will be

$$\tau = \frac{4\sqrt{n}}{p} \frac{1}{x_0^{n-1}} \int_0^{x_0} \frac{d\left(\frac{x}{x_0}\right)}{\sqrt{1 - \left(\frac{x}{x_0}\right)^{2n}}}. \tag{49}$$

The magnitude of the integral in this equation depends on the value of n and it can be concluded from eq. (49) that only for $n = 1$, i.e., for simple harmonic motion the period does not depend on the initial displacement x_0. For $n = 2$, we have

$$\int_0^{x_0} \frac{d\left(\frac{x}{x_0}\right)}{\sqrt{1 - \left(\frac{x}{x_0}\right)^4}} = \int_0^1 \frac{du}{\sqrt{1 - u^4}} = 1.31.$$

Substituting in (49)

$$\tau = 5.24 \frac{\sqrt{2}}{p} \cdot \frac{1}{x_0},$$

i.e., the period of vibration is inversely proportional to the amplitude. Such vibrations we have, for instance, in the case represented in Fig. 43, (see p. 69) if the initial tension S in the wire be equal to zero.

In a more general case when

$$f(x) = ax + bx^2 + cx^3$$

a solution of eq. 48 can be obtained by using elliptic functions.* But these solutions are complicated and not suitable for technical applications. Therefore now some graphical and numerical methods for solving eq. (48) will be developed.

19. Graphical Solution.—In the solution of the general equation (48)

* Some examples of this kind are discussed in the book "Erzwungene Schwingungen bei veränderlicher Eigenfrequenz," by G. Düffing, Braunschweig, 1918.

of pseudoharmonic vibration two integrations, shown in eq. (*b*) and (*e*) of the previous paragraph must be performed. It is only in the simplest cases that an exact integration of these is possible, but an approximate graphical solution can always be obtained on the basis of which the period of free vibration for any amplitude can be calculated with sufficient accuracy.

Let the curve *om* (Fig. 50) represent to a certain scale the restoring

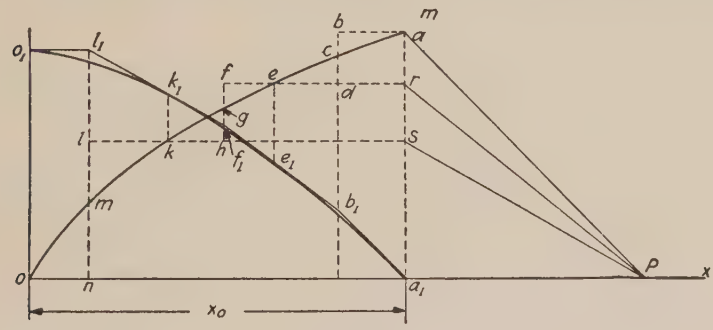


FIG. 50.

force as a function of the displacement x of the system from its middle position. From eq. (*b*) (p. 72) it is seen that by plotting the integral curve to the curve *om* the magnitude of $\dot{x}^2$ as a function of the displacement of x will be obtained. This graphical integration can be performed as follows: The continuous curve *om* is replaced by a step curve *abdfhlno* in such a manner as to make $\triangle abc = \triangle cde$, $\triangle efg = \triangle ghk$ and $\triangle klm = \triangle mno$ so that the area included between the *abdfhln* line and the x axis becomes equal to that between the *om* curve and the x axis.

A pole distance Pa_1 is now chosen such that it represents unity on the same scale as the ordinates of the *om* curve and the rays Pa, Pr, Ps are drawn. Making now $a_1b_1 \parallel Pa$, $b_1f_1 \parallel Pr$, $f_1l_1 \parallel Ps$ and $l_1o_1 \parallel Pa_1$, the polygon $a_1b_1f_1l_1o_1$ will be obtained, the slopes of whose sides are equal to the corresponding values of the function represented by *abdfhln*. This means that the $a_1b_1f_1l_1o_1$ line is the integral curve for the *abdfhln* line. Due to the equality of shaded triangles (see Fig. 50) mentioned above, the sides of the polygon $a_1b_1f_1l_1o_1$ must be tangent to the integral curve of *om*; the points of tangency being at a_1, e_1, k_1 and o_1. Therefore the curve $a_1e_1k_1o_1$ tangent to the polygon $a_1b_1f_1l_1o_1$ at a_1, e_1, k_1 and o_1 represents the integral curve for the curve *om* and gives to a certain scale the variation of the kinetic energy of the system during the motion

from the extreme position ($x = x_0$) to the middle position ($x = 0$). If the ordinates of the curve *om* are equal to a certain scale, to $2p^2f(x)$ (see eq. (*b*), p. 72) and the pole distance Pa_1 is equal to unity to the same scale then the ordinates of the $a_1e_1k_1o_1$ curve, if measured to the same scale as the displacement x_0, give the magnitudes of $\dot{x}^2$. From this the velocity $\dot{x}$ and the inverse quantity $1/\dot{x}$ can easily be calculated and the curve *pn* representing $1/\dot{x}$ as a function of x can be plotted (see Fig. 51).

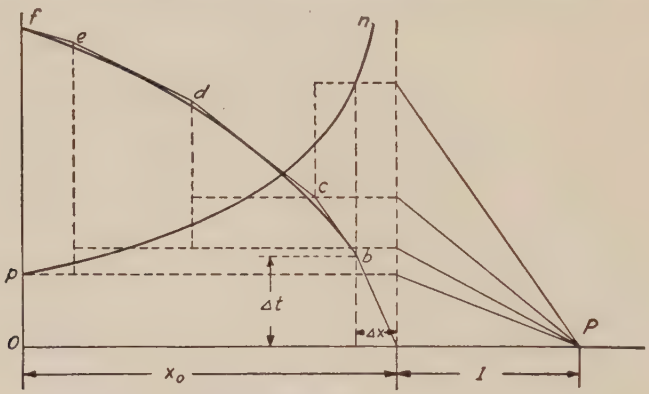


FIG. 51.

The time which will be taken by the system to reach its middle position ($x = 0$) from its extreme position ($x = x_0$) will be represented by the following integral (see eq. (*e*), p. 73)

$$t = \int_{x_0}^{0} -\frac{dx}{\dot{x}}.$$

This means that t can be obtained by plotting the integral curve of the curve *pn* (see Fig. 51) exactly in the same manner as explained above. The final ordinate *of*, measured to the same scale as x_0, gives the time t. In the case of a system symmetrical about its middle position the time t will represent a quarter of the period of free vibration for the amplitude x_0. It must be noted that for $x = x_0$, $\dot{x} = 0$, i.e., $1/\dot{x}$ becomes infinitely large at this point. In order to remove this difficulty the plotting of the integral curve can be commenced from a certain point *b*, the small coordinates x and t of which will be determined on the assumption that at the beginning along a small distance x the system moves with a constant acceleration equal to $p^2f(x)$, (see eq. (48), p. 72). Then

$$\Delta x = \frac{\Delta t^2}{2} p^2 f(x_0)$$

and

$$\triangle t = \sqrt{\frac{2\triangle x}{p^2 f(x_0)}}.$$

Another graphical method, developed by Lord Kelvin,* also can be used in discussing the differential equation of free pseudoharmonic vibration. For the general case the differential equation of motion can be presented in the following form

$$\ddot{x} = f(x, t, \dot{x}). \tag{50}$$

The solution of this equation will represent the displacement x as a function of the time t. This function can be represented graphically by a time-displacement curve (see Fig. 52). In order to obtain a definite solution the initial conditions, i.e., the initial displacement and initial velocity of the system must be known.

Let $x = x_0$ and $\dot{x} = \dot{x}_0$ for $t = 0$. Then the initial ordinate and initial slope of the time-displacement curve will be known. Substituting the initial values of x and $\dot{x}$ in eq. (50), the initial value of $\ddot{x}$ can be calculated. Now from the known equation,

$$\rho = \frac{\sqrt{(1 + \dot{x}^2)^3}}{\ddot{x}}, \tag{a}$$

the radius of curvature ρ_0 at the beginning of the time-displacement curve will be found. By using this radius a small element a_0a_1 of the time-displacement curve can be traced as an arc of a circle (Fig. 52) and the values of the ordinate $x = x_1$ and of the slope $\dot{x} = \dot{x}_1$ at the new point a_1 can be taken from the drawing and the corresponding value of $\ddot{x}$ calculated from eq. (50). Now from eq. (a) the magnitude of $\rho = \rho_1$ will be obtained by the use of which the next element a_1a_2 of the curve can be traced. Continuing this construction, as described, the time-displacement curve will be graphically obtained. The calculations

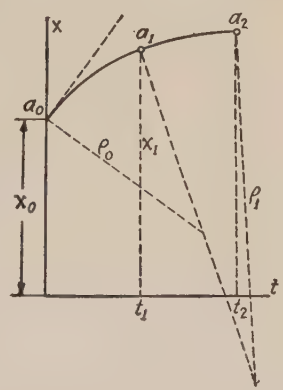


FIG. 52.

* See, Lord Kelvin, On Graphic Solution of Dynamical Problems, Phil. Mag., Vol. 34 (1892). The description of this and several other graphical methods of integrating differential equations can be found in the book "Die Differentialgleichungen des Ingenieurs," by W. Hort (2d ed., 1925), Berlin, which contains applications of these methods to the solution of technical problems. See also H. von Sanden, Practical Mathematical Analysis, New York, 1926.

involved can be somewhat simplified by using the angle of inclination of a tangent to the time-displacement curve. Let θ denote this angle, then

$$\dot{x} = \tan\theta \qquad \text{and} \qquad \ddot{x} = f(x, t, \tan\theta).$$

Substituting in eq. (a)

$$\rho = \frac{\sqrt{(1 + \tan^2\theta)^3}}{f(x, t, \tan\theta)} = \frac{1}{\cos^3\theta\, f(x, t, \tan\theta)}. \tag{b}$$

In this calculation the square root is taken with the positive sign so that the sign of ρ is the same as the sign of $\ddot{x}$. If $\ddot{x}$ is negative the center of curvature must be taken in such a manner as to obtain the curve convex up (see Fig. 52).

In the case of free vibration and by neglecting damping, eq. (50) assumes the form given in (48) and the graphical integration described above becomes very simple, because the function f depends in this case only on the magnitude of displacement x. Taking for the initial conditions $x = x_0$ and $\dot{x} = 0$ for $t = 0$, the time-displacement curve will have the general form shown in Fig. 53. In the case of a system symmetrical about the middle position the intersection of this curve with the t axis will determine the period τ of the free vibration of the system. The magnitude of τ can always be determined in this manner with an accuracy sufficient for practical applications. In Fig. 53 for instance, the case of a simple harmonic vibration was taken for which the differential equation is

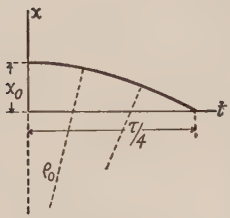


Fig. 53.

$$\ddot{x} + p^2x = 0$$

and the exact solution gives

$$\tau = \frac{2\pi}{p}.$$

Equation (b) for this case becomes

$$\rho = \frac{1}{\cos^3\theta \cdot p^2x}. \tag{b^1}$$

The initial displacement x_0 in Fig. 53 is taken equal to 20 units of length and ρ_0 equal to 100 units of length. Then from eq. (b^1) for $\theta = 0$, we obtain

$$\frac{1}{p} = \sqrt{20 \cdot 100} = 44.7 \text{ units}. \tag{c}$$

The quantity $1/p$ has the dimension of time and the length given by eq. (c) should be used in determining the period from Fig. 53. By measuring, we obtain from this figure

$$\frac{\tau}{4} = 69.5 \text{ units}$$

or by using (c)

$$\tau = 1/p \cdot \frac{4 \times 69.5}{44.7} = \frac{6.22}{p}.$$

In the graphical solution only 7 intervals have been taken and the result obtained is accurate within 1%.

20. Numerical Solution.—Pseudoharmonic vibrations as given by equations (48) and (50) can also be solved in a numerical way. Consider as an example free vibration without damping. The corresponding differential equation is

$$\ddot{x} + p^2 f(x) = 0. \qquad (a)$$

Let the initial conditions be

$$x = x_0; \qquad \dot{x} = 0, \qquad \text{for} \qquad t = 0. \qquad (b)$$

By substituting x_0 for x in eq. (a) the magnitude of $\ddot{x}_0$ can be calculated. By using the value $\ddot{x}_0$ of the acceleration at $t = 0$ the magnitude of $\dot{x}_1$ and x_1, i.e., the velocity and displacement at any moment t_1 chosen very close to the time $t = 0$ can be calculated. Let Δt denote the small interval of time between the moment $t = 0$ and the moment $t = t_1$. The approximate values of $\dot{x}_1$ and x_1 will then be obtained from the following equations,

$$\dot{x}_1 = \dot{x}_0 + \ddot{x}_0 \Delta t; \qquad x_1 = x_0 + \frac{\dot{x}_0 + \dot{x}_1}{2} \Delta t. \qquad (c)$$

Substituting the value x_1 for x in eq. (a), the value of $\ddot{x}_1$ will be obtained. By using this latter value better approximations for $\dot{x}_1$ and x_1 can be calculated from the following equations,

$$\dot{x}_1 = \dot{x}_0 + \frac{\ddot{x}_0 + \ddot{x}_1}{2} \cdot \Delta t \qquad \text{and} \qquad x_1 = x_0 + \frac{\dot{x}_0 + \dot{x}_1}{2} \Delta t. \qquad (d)$$

A still better approximation for $\ddot{x}_1$ will now be obtained by substituting the second approximation of x_1 (eq. (d)) in eq. (a). Now, taking the second step, by using x_1, $\dot{x}_1$ and $\ddot{x}_1$ the magnitude of x_2, $\dot{x}_2$, $\ddot{x}_2$ for the time $t = t_2 = 2\Delta t$ can be calculated exactly in the same manner as explained above. By taking the intervals Δt small enough and making the calculations for every value of t twice as explained before in order to obtain

the second approximation, this method of numerical integration can always be made sufficiently accurate for practical applications.*

In order to show this procedure of calculation and to give some idea of the accuracy of the method we will consider the case of simple harmonic vibration, for which the equation of motion is:

$$\ddot{x} + p^2x = 0.$$

The exact solution of this for the initial conditions (*b*) is

$$x = x_0 \cos pt; \qquad \dot{x} = -x_0 p \sin pt. \tag{e}$$

The results of the numerical integration are given in the table below. The length of the time intervals was taken equal to $\Delta t = 1/4p$. Remembering that the period of vibration in this case is $\tau = 2\pi/p$ it is seen that Δt, the interval chosen, is equal approximately to 1/6 of a quarter of the period τ. The second line of the table expresses the initial conditions. Now, for obtaining first approximations for $\dot{x}_1$ and x_1, at the time $t = \Delta t = 1/4p$, equations (*c*) were used. The results obtained are given in the third line of the table. For getting better approximations for $\dot{x}_1$ and x_1, equations (*d*) were used and the results are

TABLE I

NUMERICAL INTEGRATION

t	x	$\dot{x}$	$\ddot{x}$	$\cos pt$	$\sin pt$
0	x_0	0	$-p^2x_0$	1	0
Δt	.9687 x_0	− .2500 px_0	−.9687 p^2x_0		
Δt	.9692 "	− .2461 "	−.9692 "	.9689	.2474
$2\Delta t$	.8774 "	− .4884 "	−.8774 "		
$2\Delta t$	.8788 "	− .4769 "	−.8788 "	.8776	.4794
$3\Delta t$	.7321 "	− .6966 "	−.7321 "		
$3\Delta t$	.7344 "	− .6783 "	−.7344 "	.7317	.6816
$4\Delta t$	.5419 "	− .8619 "	−.5419 "		
$4\Delta t$	.5449 "	− .8378 "	−.5449 "	.5403	.8415
$5\Delta t$	.3184 "	− .9740 "	−.3184 "		
$5\Delta t$	.3220 "	− .9457 "	−.3220 "	.3153	.9490
$6\Delta t$	.0755 "	−1.0262 "	−.0755 "		
$6\Delta t$	.0794 "	− .9954 "	−.0794 "	.0707	.9975
$7\Delta t$	−.1719 "	−1.0153 "	−.1719 "		
$7\Delta t$	−.1680 "	− .9838 "	−.1680 "	−.1782	.9840

* The method can also be used in the case of a more general type of equation, for instance, such as eq. 50 (see p. 77). It was used by the writer in the solution of the problem on stress distribution in rotating discs (see Bulletin of the Institute of Technology, St. Petersburg, 1912) and proved to be satisfactory.

put in the fourth line of the table. Proceeding in this manner the complete table was calculated. In the last two columns the corresponding values of *sin pt* and *cos pt* proportional to the exact solutions (*e*) are given, so that the accuracy of the numerical integration can be seen directly from the table. We see that the velocities obtained by calculation have always a high accuracy. The largest error in the displacement is seen from the last line of the table and amounts to about 1% of the initial displacement x_0.

These results were obtained by taking only 6 intervals in a quarter of a period. By increasing the number of intervals the accuracy can be increased, but at the same time the number of necessary calculations becomes larger.

By using the table the period of vibration also can be calculated. It is seen from the first and second columns that for $t = 6\Delta t$ the time-displacement curve has a positive ordinate equal to $.0794x_0$. For $t = 7\Delta t$ the ordinate of the same curve is negative and equal to $.1680x_0$. The point of intersection of the time-displacement curve with the t axis determines the time equal to a quarter of the period of vibration. By using linear interpolation this time will be found from the equation

$$\frac{1}{4}\tau = 6\Delta t + \Delta t\frac{.0794}{.0794 + .1680} = 6.32\Delta t = \frac{6.32}{4p} = \frac{1.58}{p}.$$

The exact value of the quarter of a period of vibration is $\pi/2p = 1.57/p$. It is seen that by the calculation indicated the period of vibration is obtained with an error less than 1%. From this example it is easy to see that the numerical method described can be very useful for calculating the period of vibration of systems having a flexibility which varies with the displacement.

21. Forced Pseudoharmonic Vibrations.—Assume now that a system, the spring constant of which changes with the displacement, is submitted to the action of a periodical disturbing force proportional to *sin mt*. Then, instead of eq. (48) the following equation holds for the forced vibrations:

$$\ddot{x} + p^2 f(x) = a \sin mt. \tag{51}$$

In the following the particular case of a system, for which eq. (51) becomes

$$\ddot{x} + p^2 x - \gamma x^3 = a \sin mt \tag{52}$$

will be considered. This system is symmetrical about its middle position as can be seen from the fact that by changing x to $-x$ the magnitude

of the restoring force does not change. The free vibrations of this system can be studied by one of the methods described above and it can be shown that if $\gamma > 0$, the period of vibration of the system increases with an increase in amplitude. It must be noted that eq. (52) is not linear, therefore its general solution cannot be obtained by superimposing the forced vibration, represented by a particular solution of eq. (52), on the free vibrations. For the same reason, if, instead of a sinusoidal force, $a \sin mt$, a more complicated force represented by a Fourier series, were acting on the system, the resultant motion cannot be obtained by simple superposition of the motions produced by the individual sinusoidal terms of the series.

A general solution of eq. (52) has not been established so that some approximate method in studying the forced vibrations must be applied. The graphical and numerical methods used above for investigating the free vibrations can also be used here, but the step-by-step integration is too laborious and not accurate enough if several cycles are to be investigated in order to establish the variation in amplitude of the forced vibration with time. A more satisfactory result will be obtained by using a method of successive approximations which will now be explained.

Consider first the case of simple harmonic motion, i.e., γ in eq. (52) becomes 0. The equation of motion thus becomes

$$\ddot{x} + p^2x = a \sin mt$$

or

$$\ddot{x} = -p^2x + a \sin mt. \tag{a}$$

We will not consider here the initial conditions and take as a first approximation of the solution of eq. (*a*) the form

$$x_1 = A \sin mt. \tag{b}$$

Substituting this in the right side of eq. (*a*) the following equation for calculating a second approximation will be obtained

$$\frac{d^2x_2}{dt^2} = (a - p^2A) \sin mt,$$

from which

$$x_2 = \frac{Ap^2 - a}{m^2} \sin mt. \tag{c}$$

The arbitrary constant A will now be chosen in such a manner so as to make the amplitudes of the vibrations x_1 and x_2 equal, then

$$A = \frac{Ap^2 - a}{m^2},$$

and we obtain

$$A = \frac{a}{p^2 - m^2}. \tag{d}$$

This coincides with the exact solution for the case of forced harmonic vibration (see eq. (19), p. 9).

Proceeding in the same manner in the general case of eq. (52), ($\gamma \neq 0$), let

$$x_1 = A \sin mt$$

represent the first approximation of the solution of this equation. Then for calculating the second approximation the following equation (see eq. (52)) will be obtained:

$$\frac{d^2x_2}{dt^2} = -p^2x_1 + \gamma x_1^3 + a \sin mt = (a - Ap^2) \sin mt + A^3\gamma \sin^3 mt. \tag{e}$$

Taking into consideration that

$$\sin^3 mt = \tfrac{1}{4}(3 \sin mt - \sin 3\, mt)$$

we obtain by integrating eq. (*e*):

$$x_2 = \frac{1}{m^2}(Ap^2 - a - \tfrac{3}{4}A^3\gamma) \sin mt + \frac{A^3\gamma}{36m^2} \sin 3\, mt.$$

Now we determine the arbitrary constant A so as to make the amplitude of the fundamental vibration in the first and the second approximations equal, then

$$\frac{1}{m^2}(Ap^2 - a - \tfrac{3}{4}A^3\gamma) = A$$

and we obtain

$$a + A(m^2 - p^2) = -\tfrac{3}{4}A^3\gamma. \tag{53}$$

The second approximation will now have the form

$$x_2 = A \sin mt + \frac{A^3\gamma}{36m^2} \sin 3\, mt, \tag{54}$$

in which A is the root of the cubic eq. (53).

By using this result, the third approximation can be obtained from eq. (52) and so on. This method of successive approximation is based on the assumption that the quantity $(\gamma/m^2)A^2$ is small in comparison with 1, but a more detailed investigation of the convergency of this

process of calculation shows * that the approximate solution (54) is always sufficiently accurate for technical applications.

For discussing the roots of the cubic equation (53) a graphical method is very useful. Presenting this equation in the form

$$A\left(1-\frac{m^2}{p^2}\right)-\frac{a}{p^2}=\tfrac{3}{4}\frac{\gamma}{p^2}A^3, \qquad (53)^1$$

we obtain the roots as the abscissas of the points of intersection of the straight line

$$y=A\left(1-\frac{m^2}{p^2}\right)-\frac{a}{p^2} \qquad (f)$$

with the cubic parabola

$$y=\tfrac{3}{4}\frac{\gamma}{p^2}A^3. \qquad (g)$$

In this manner Fig. 54 has been obtained.

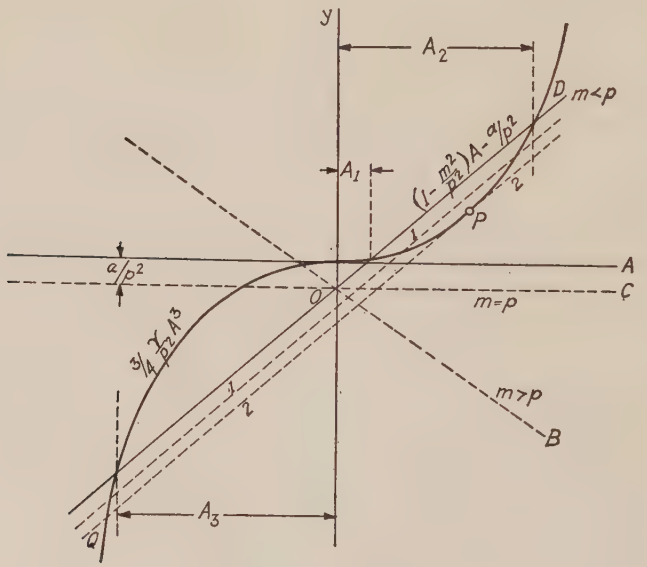


Fig. 54.

* The problem of free and forced pseudoharmonic vibrations is discussed in the book: "Erzwungene Schwingungen bei veränderlicher Eigenfrequenz," Braunschweig, 1918, by Georg Duffing. See also the paper: Einige unharmonische Schwingungsformen mit grosser Amplitude by R. Rüdenberg, Zeitschr. f. Angew. Math. und Mech., Vol. 3, p. 454 (1923).

Three straight lines, OB, OC and OD, are drawn which correspond to three different relations between m and p in eq. (f) for the same value of a/p^2. If a vibration of very small amplitude takes place, the displacement x is very small and the term containing x^3 in eq. (52) can be neglected. In such a manner a simple harmonic motion of the period $\tau = 2\pi/p$ will be obtained. The straight line OB in Fig. 54 represents the condition, when the frequency of the disturbing force is higher than the frequency of this harmonic motion ($m > p$). It is seen from the figure that in this case we have only one point of intersection which determines completely the amplitude of the forced vibration. This amplitude decreases with an increase in the frequency of the disturbing force, i.e., with increasing ratio m/p.

When $m = p$, also only one real solution is possible. The corresponding amplitude of vibration will be again obtained as the abscissa of the point of intersection of the line OC with the cubic parabola.

Consider now the case when $m < p$. Let OD represent for this case the straight line, given by eq. (f). The point of intersection of this line with the A axis will be found from the equation

$$\left(1 - \frac{m^2}{p^2}\right) A - \frac{a}{p^2} = 0,$$

from which

$$A = \frac{a}{p^2 - m^2}.$$

This is the amplitude of a simple harmonic motion (see eq. (d)) and in the case of small amplitudes it differs only a little from the magnitude of the root A_1 of the cubic eq. (53). This means that if a very small disturbing force is acting, vibrations will be built up, which differ only slightly from a simple harmonic motion. Imagine now that we increase gradually the magnitude of the disturbing force, i.e., the quantity a, without changing its frequency. The line OD is thus displaced in a downward direction, parallel to itself, as for example, position 1–1 in Fig. 52. The amplitude of the forced vibration also increases. It is seen from the figure that by gradually increasing the force we arrive finally at the limiting condition represented by the line 2–2, tangent to the positive branch of the cubic parabola. Beyond this limiting condition the roots A_1 and A_2 become imaginary and the amplitude of the forced vibration will then be determined by the third root A_3, corresponding to the intersection of the straight line with the negative branch of the cubic parabola.

Experiments made * show that by a gradual increase in the disturbing force vibrations corresponding to the root A_1 (Fig. 54) are built up. Beyond the limiting condition, represented by the line 2–2, the vibrations corresponding to the root A_1 are damped out and the new type of vibration with a phase lag equal to π (180°) and an amplitude, represented by the root A_3, begins.

It must be noted that the graphical method described above can be used also in discussing free pseudoharmonic vibrations. It is only necessary to put $a = 0$ in our previous investigation. Then the point O in Fig. 54 will coincide with the origin of coordinates. For any frequency of vibration, $(0 < m < p)$ which is possible for the system being considered, a straight line

$$y = A\left(1 - \frac{m^2}{p^2}\right)$$

can be constructed. The abscissa of the point of intersection of this line with the cubic parabola (eq. (g)) will now determine the corresponding amplitude of free vibration of the system.

From this discussion it will be appreciated that by using the method of successive approximations a solution of the equation for forced pseudoharmonic vibrations can be obtained, which has the same period as the disturbing force. The amplitude of this vibration always remains *finite* and a condition, known as "resonance" in the case of harmonic vibrations, does not occur in systems with variable flexibility. If for such systems the term "resonance" be applied to the case when $m = p$, then it can be stated that for this case and for those where the frequency of the disturbing force is larger than that of "resonance," the amplitude of vibration has a definite magnitude. For the case $m < p$, a great change in the amplitude and in the lag of the *forced* vibration may under certain conditions take place, corresponding to a sudden change from P to Q in Fig. 54. In order to obtain a stable condition and remove the possibility of a large variation in the amplitude of the forced vibration, m should either be taken greater than p or several times less than p. It can be seen from Fig. 54 that when m is very different from p the points of intersection determining the amplitude of the forced vibration will be

* Very important experimental investigations of pseudoharmonic vibrations were made by O. Martienssen (Phys. Zeitschr., 1910, p. 48). He was the first to show experimentally that in the case of pseudoharmonic vibrations the same disturbing force may produce forced vibrations of two distinct and different amplitudes. Some experiments on pseudoharmonic forced vibration by using a pendulum are described on p. 76 of the book by G. Duffing, mentioned above.

situated on the flat part of the cubic parabola and the type of motion differs only slightly from the simple harmonic vibration.

22. Quasiharmonic Vibrations.—There are certain technical problems in which the vibrating system has a spring characteristic which depends on the time. The motion for such systems is called quasiharmonic and can be represented by the equation

$$\ddot{x} + 2n\dot{x} + p^2x = f(t),$$

where the spring characteristic p^2 or damping factor $2n$ are periodic functions of the time. Several examples of such vibrations will now be discussed and the problem of the vibrations of a certain type of electric locomotive will be considered in detail.

As a simple example the oscillations of a pendulum of variable length l (Fig. 55) will be considered. By pulling the string OA with a force T, a variation in the length l of the pendulum can be produced. In order to obtain the differential equation of motion the principle of angular momentum will be applied. The momentum of the moving mass W/g can be resolved into two components, one in the direction of the string OA and another in the direction perpendicular to OA. In calculating the angular momentum about the point O only the second component equal to $(W/g)l\dot{\theta}$, must be taken into consideration. The derivative of this angular momentum with respect to the time t should be equal to the moment of the forces acting about the point O. Hence the equation

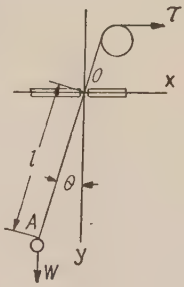


FIG. 55.

$$\frac{d}{dt}\left(\frac{W}{g}l^2\dot{\theta}\right) = -Wl\sin\theta,$$

or

$$\ddot{\theta} + \frac{2}{l}\frac{dl}{dt}\dot{\theta} + \frac{g}{l}\sin\theta = 0. \tag{55}$$

In the case of vibrations of small amplitude, θ can be substituted for $\sin\theta$ in eq. (55) and we obtain

$$\ddot{\theta} + \frac{2}{l}\frac{dl}{dt}\dot{\theta} + \frac{g}{l}\theta = 0. \tag{55^1}$$

Comparing this equation with eq. (25) (see p. 23) for damped vibration, we see that the term containing the derivative dl/dt takes the place of the

term representing the damping in eq. (25). By appropriate variation of the length l with time the same effect can be produced as with "*negative damping.*" In such a case a progressive accumulation of energy in the system instead of a dissipation of energy takes place and the amplitude of the oscillation of the pendulum increases with the time. It is easy to see that such an accumulation of energy results from the work done by the tensile force T during the variation in the length l of the pendulum. Various methods of varying the length l can be imagined which will result in the accumulation of energy of the vibrating system.

As an example consider the case represented in Fig. 56 in which the

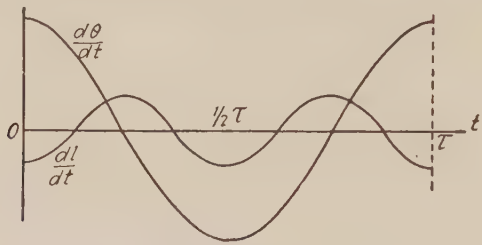


Fig. 56.

angular velocity $d\theta/dt$ of the pendulum and the velocity dl/dt of variation in length of the pendulum are represented as functions of the time. The period of variation of the length of the pendulum is taken half that of the vibration of the pendulum and the $d\theta/dt$ line is situated in such a manner with respect to the dl/dt line that the maximum negative damping effect coincides with the maximum speed. This means that a decrease in the length l has to be produced while the velocity $d\theta/dt$ is large and an increase in length l while the velocity is comparatively small. Remembering that the tensile force T is working against the radial component of the weight W together with the centrifugal force, it is easy to see that in the case represented in Fig. 56 the work done by the force T during any decrease in length l will be larger than that returned during the increase in length l. The surplus of this work results in an increase in energy of vibration of the pendulum.

The calculation of the increase in energy of the oscillating pendulum becomes especially simple in the case shown in Fig. 57. It is assumed in this case that the length of the pendulum is suddenly decreased by the quantity Δl when the pendulum is in its middle position and is suddenly increased to the same amount when the pendulum is in its extreme positions. The trajectory of the mass W/g is shown in the figure by the

full line. The mass performs two complete oscillations during one oscillation of the pendulum. The work produced during the shortening of the length l of the pendulum will be

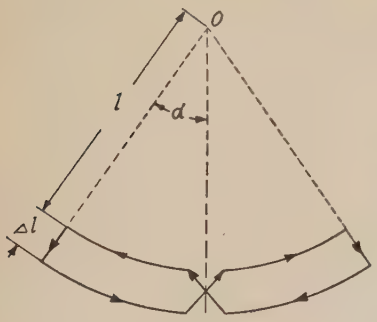


FIG. 57.

$$\left(W + \frac{W}{g}\frac{v^2}{l}\right)\Delta l.^* \qquad (a)$$

Here v denotes the velocity of the mass W/g of the pendulum when in its middle position. The work returned at the extreme positions of the pendulum is

$$W\Delta l \cos \alpha. \qquad (b)$$

The gain in energy during one complete oscillation of the pendulum will be

$$\Delta E = 2\left\{\left(W + \frac{W}{g}\frac{v^2}{l}\right)\Delta l - W\Delta l \cos \alpha\right\},$$

or by putting

$$v^2 = 2gl(1 - \cos \alpha),$$

we have

$$\Delta E = 6W\Delta l(1 - \cos \alpha). \qquad (c)$$

Due to this increase in energy a progressive increase in amplitude of oscillation of the pendulum takes place in the case shown in Fig. 57.

As another example consider the flexural vibrations of a rotating bar carrying a load W at its end (Fig. 58). This case represents the usual

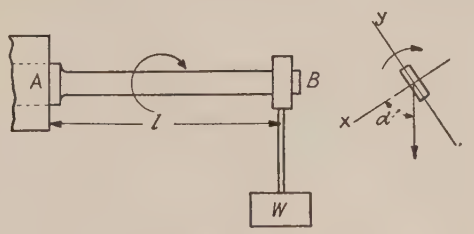


FIG. 58.

arrangement of the cantilever beam fatigue testing machine. If the flexural rigidity of the bar is the same in all axial sections the deflection at the end B remains constant during rotation and no vibrations will take place.

* In this calculation the variation in centrifugal force during the shortening of the pendulum is neglected.

Assume now that the principal moments of inertia I_x and I_y of the cross section of the bar are different (Fig. 58, *b*). Then the statical vertical deflection of the end B of the bar will depend on the angle α between the direction of the load and the x axis and can be represented in the following form:

$$\frac{Wl^3}{6EI_x}\left\{1 + \frac{I_x}{I_y} - \left(1 - \frac{I_x}{I_y}\right)\cos 2\alpha\right\}. \qquad (d)$$

Due to this variation in vertical deflection the frequency of the free vertical vibration of the load W, attached to the end B of the bar AB, will depend on the magnitude of the angle α.

Assume now that the bar AB rotates with a constant speed ω. Then we can put $\alpha = \omega t$ in eq. (*d*), and the spring characteristic in the case of vertical vibration of the load W becomes (from (*d*)):

$$k = \frac{6EI_x}{\left[1 + \frac{I_x}{I_y} - \left(1 - \frac{I_x}{I_y}\right)\cos 2\omega t\right]l^3}, \qquad (e)$$

i.e., the spring characteristic is a periodic function of the time frequency twice as large as the frequency of rotation of the bar. Several tests were made with such an arrangement in which $I_y = .84\ I_x$ and it was found that two critical speeds of rotation can be established at which large vertical vibrations of the load W take place. The result of one of these experiments is shown in Fig. 59. In this

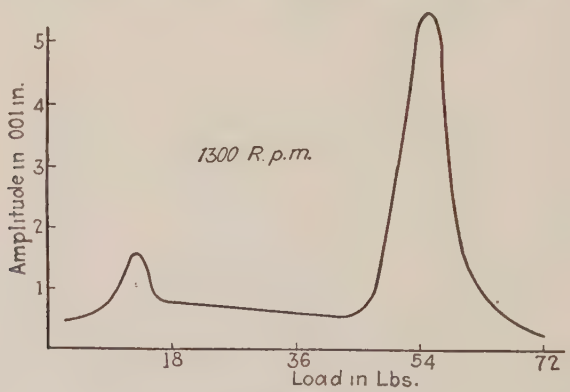


Fig. 59.

figure the amplitude of vibration is plotted as a function of the weight W, the bar being rotated at a constant speed of 1300 R.P.M. Two critical

conditions were obtained. The vibration with the larger weight corresponds to the natural frequency of 1300 cycles per minute of vertical vibration of the weight attached to the bar (taking for moment of inertia of the bar the average of the two extreme values). In this case there are two full cycles of change in flexibility per one cycle of the vibration. The other critical region W_c (at 1/4 of the previous weight) corresponds to a natural frequency of the weight of 2600 cycles per minute. In this case one cycle of the change in flexibility corresponded to one cycle of the vibration.

Analogous kinds of vibration may occur in a rotor having a variable flexural rigidity, for instance, in a two pole rotor of a turbo generator. The deflection of such a rotor under the action of its own weight varies during rotation and at a certain speed heavy vibration, due to this variable flexibility, may take place.

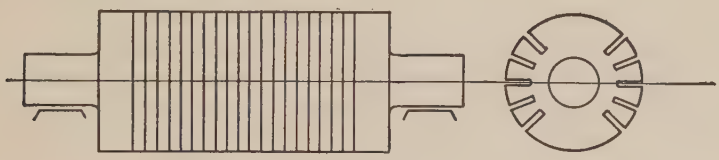


Fig. 60.

23. Vibrations in the Side Rod Drive System of Electric Locomotives.—*General.*—One of the most important technical examples of quasiharmonic vibrations is to be found in the case of electric locomotives with side rod drive. The flexibility of the system between the motor shaft and the driving axles depends on the position of the cranks and during uniform motion of the locomotive this is usually a complicated function, the period of which corresponds to one revolution of the driving axles. We have seen in the previous paragraph that such systems of variable flexibility under certain conditions may be brought into heavy vibrations. Due to the fact that such vibrations are accompanied by a fluctuation in the angular velocity of the heavy rotating masses of the motors, large additional dynamical forces will be produced in the driving system of the locomotive. Many failures especially in the early period of electric locomotive building must be attributed to this dynamical cause.*

* The most important papers dealing with vibrations in electric locomotives are:

1. "Ueber Schüttelerscheinungen in Systemen mit periodisch veränderlicher Elastizität," by Prof. E. Meissner, Schweizerische Bauzeitung, Vol. 72 (1918), p. 95.
2. "Ueber die Schüttelschwingungen des Kuppelstangantriebes," by K. E. Müller, Schweizerische Bauzeitung, Vol. 74 (1919), p. 141.

(over)

Variable Flexibility of Side Rod Drive.—In order to show how the flexibility of a side rod drive changes during rotation of the motor a simple example shown in Fig. 61 will now be considered. A torque

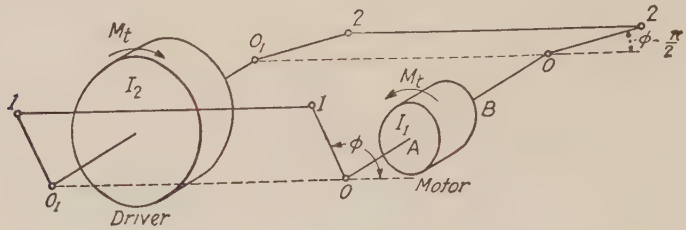


FIG. 61.

moment M_T, acting on the rotor, is transmitted to the driving axle $0_1 0_1$ through the motor shaft 00, cranks 01 and 02 and side rods 11 and 22. Consider now the angle of rotation of the rotor with respect to the driving axle $0_1 0_1$ due to twist of the shaft 00 and due to deformation of the side rods. Let M'_T and M''_T be the torque moments transmitted to the driving axle through the side rods 11 and 22, respectively, then:

$$M_T = M'_T + M''_T \tag{a}$$

and if K_1 = spring characteristic for the end OA of the shaft, then the angle of rotation of the motor due to twist of the shaft will be given by

$$\Delta_1\varphi = \frac{M'_T}{K_1}. \tag{b}$$

Consider now the angle of rotation $\Delta_2\varphi$ due to compression of the side rod 11. Let,

S_1 = compressive force in this side rod

$\delta = \dfrac{S_1 l}{AE}$ = the corresponding compression of the side rod,

r = the radius of the crank.

Then we have

$$M'_T = S_1 r \sin \varphi \tag{c}$$

3. "Eigenschwingungen von Systemen mit periodisch veränderlicher Elastizität," by L. Dreyfus. "A. Föppl zum siebzigsten Geburtstag" (1924), p. 89.
4. A. Wichert, Schüttelerscheinungen, Forschungsarbeiten, Heft 266 (1924).
5. E. E. Seefehlner, Elektrische Zugförderung, 1924.

and from a geometrical consideration (see Fig. 62),

$$\delta = r\Delta_2\varphi \sin \varphi. \tag{d}$$

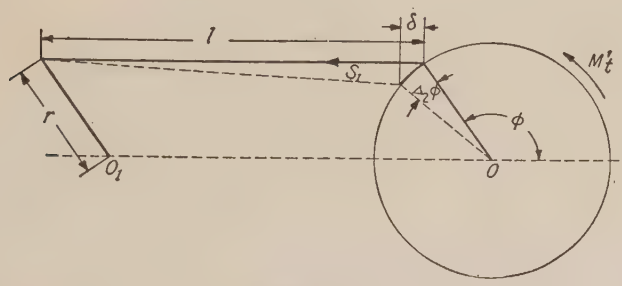


FIG. 62.

Remembering that

$$S_1 = \delta \frac{AE}{l},$$

we have from eq. (*c*) and (*d*),

$$\Delta_2\varphi = \frac{M'_T l}{AEr^2 \sin^2 \varphi}$$

or, by letting

$$\frac{AEr^2}{l} = K_2$$

we have,

$$\Delta_2\varphi = \frac{M'_T}{K_2 \sin^2 \varphi}. \tag{e}$$

The complete angle of rotation of the motor with respect to the driving axle will be *

$$\Delta\varphi = \Delta_1\varphi + \Delta_2\varphi = M'_T\left(\frac{1}{K_1} + \frac{1}{K_2 \sin^2 \varphi}\right). \tag{f}$$

The same angle should be obtained from a consideration of the twist of the end *OB* of the shaft and of compression of the side rod 22. Assuming that the arrangement is symmetrical about the longitudinal axis of the locomotive and repeating the same reasoning as above we obtain,

$$\Delta\varphi = M''_T\left(\frac{1}{K_1} + \frac{1}{K_2 \cos^2 \varphi}\right). \tag{g}$$

* The deformation of the side rods and of the shaft 00 only are taken into consideration in this analysis.

From equations (a), (f) and (g), we have

$$M_T = \Delta\varphi \frac{\dfrac{2}{K_1}\sin^2\varphi\cos^2\varphi + \dfrac{1}{K_2}}{\left(\dfrac{1}{K_1}\sin^2\varphi + \dfrac{1}{K_2}\right)\left(\dfrac{1}{K_1}\cos^2\varphi + \dfrac{1}{K_2}\right)}$$

$$= \Delta\varphi \frac{\dfrac{2}{K_1} + \dfrac{8}{K_2} - \dfrac{2}{K_1}\cos 4\varphi}{\dfrac{8}{K_2^2} + \dfrac{8}{K_1K_2} + \dfrac{1}{K_1^2} - \dfrac{1}{K_1^2}\cos 4\varphi}. \qquad (h)$$

Putting

$$\varphi = \omega t, \qquad \frac{2}{K_1} + \frac{8}{K_2} = a, \qquad \frac{2}{K_1} = b,$$

$$\frac{8}{K_2^2} + \frac{8}{K_1K_2} + \frac{1}{K_1^2} = c, \qquad \frac{1}{K_1^2} = d,$$

we have

$$M_T = \Delta\varphi \frac{a - b\cos 4\omega t}{c - d\cos 4\omega t}. \qquad (56)$$

It is seen that the flexibility of the system is a function with a period four times smaller than the period of one revolution of the shaft. In Fig. 63 the variation of the flexibility with the angle is represented

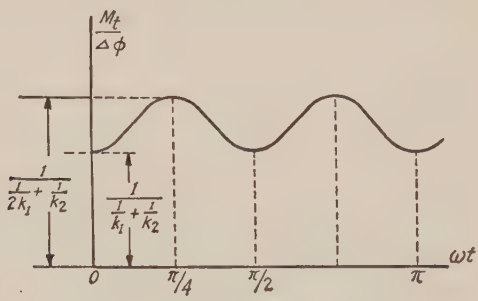


Fig. 63.

graphically. For a given value of torque moment the angle of twist becomes maximum and equal to $M_T\left(\dfrac{1}{K_1} + \dfrac{1}{K_2}\right)$ when

$$\varphi = \omega t = 0, \frac{\pi}{2}, \pi \cdots.$$

It becomes minimum and equal to $M_T\left(\frac{1}{2K_1}+\frac{1}{K_2}\right)$ when

$$\varphi = \omega t = \frac{\pi}{4}, \frac{3\pi}{4} \cdots.$$

It is easy to see that the fluctuation in the flexibility of the system decreases when the rigidity of the shaft, i.e., the quantity K_1, increases. For an absolutely rigid shaft the flexibility of the system remains constant during rotation. In our above consideration (*a*), (*f*) and (*g*) were solved analytically. The same equations can, however, be easily solved graphically.* Let AB represent to a certain scale the magnitude of the torque moment M_T (see Fig. 64); then by taking the end ordinates AF

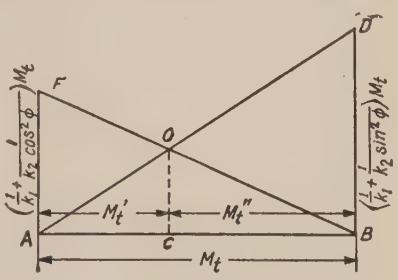


FIG. 64.

and BD equal to $\left(\frac{1}{K_1}+\frac{1}{K_2\cos^2\varphi}\right)M_T$ and $\left(\frac{1}{K_1}+\frac{1}{K_2\sin^2\varphi}\right)M_T$, respectively the vertical OC through the point of intersection of the lines BF and AD will determine AC and CB, the magnitudes of the moments M_T' and M_T''.

From the figure we have also,

$$OC = M'_T\left(\frac{1}{K_1}+\frac{1}{K_2\sin^2\varphi}\right) = \Delta\varphi,$$

i.e., OC is equal to the angle of rotation of the motor due to deformation of the side rod drive produced by the torque moment M_T.

This graphical method is especially useful for cases in which clearances, as well as elastic deformations are considered. Consider, for instance, the effect of the clearance between side rod and crankpin. Let a denote the magnitude of this *clearance*,† then the displacement (see Fig. 62) will be equal to the compression of the side rod together with the clearance a and we have,

$$\delta = \frac{S_1 l}{AE} + a$$

* This method was used by A. Wiechert, loc. cit., p. 92.

† a denotes the difference between the radius of the bore and the radius of the pin.

or, by using eq. (c) and (d)

$$r\Delta_2\varphi \sin \varphi = \frac{M'_T}{r \sin \varphi} \frac{l}{AE} + a$$

$$\Delta_2\varphi = \frac{M'_T}{K_2 \sin^2 \varphi} + \frac{a}{r \sin \varphi}.$$

The complete angle of rotation will be

$$\Delta\varphi = \Delta_1\varphi + \Delta_2\varphi = \frac{M'_T}{K_1} + \frac{M'_T}{K_2 \sin^2 \varphi} + \frac{a}{r \sin \varphi}. \qquad (k)$$

In the same manner, by considering the other crank, we obtain

$$\Delta\varphi = \frac{M''_T}{K_1} + \frac{M''_T}{K_2 \cos^2 \varphi} - \frac{a}{r \cos \varphi}. \qquad (l)$$

From equations (k), (l) and (a) the moments M'_T and M''_T and the angle $\Delta\varphi$ can be calculated. A graphical solution of these equations is shown in Fig. 65. AB represents, as before, the complete torque moment

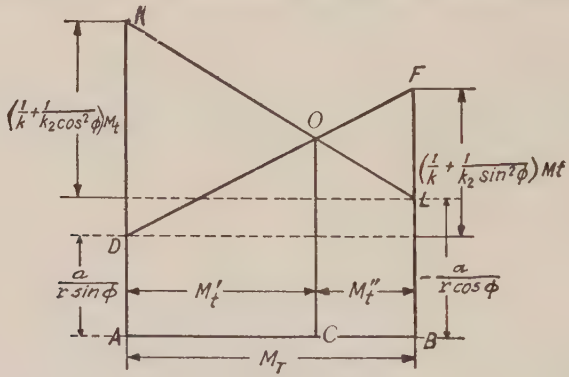


FIG. 65.

M_T. The straight lines DF and LK represent the right sides of equations (k) and (l) as linear functions of M'_T and M''_T, respectively. The point of intersection O of these two lines gives us the solution of the problem. It is easy to see that the ordinate OC is equal to the angle $\Delta\varphi$ and that the distances AC and CB are equal to the torque moments M'_T and M''_T, respectively.

It is seen from Fig. 65 that for the position of the cranks when

$$-\frac{a}{r \cos \varphi} = \frac{a}{r \sin \varphi} + \left(\frac{1}{K_1} + \frac{1}{K_2 \sin^2 \varphi}\right) M_T \qquad (m)$$

M_T'' becomes equal to 0. For smaller values of φ * than those given by eq. (m) the side rod 1–1 takes the complete torque moment and $M_T' = M_T$. In the same manner for angles larger than those obtained from the equation,

$$\frac{a}{r \sin \varphi} = -\frac{a}{r \cos \varphi} + \left(\frac{1}{K_1} + \frac{1}{K_2 \cos^2 \varphi}\right) M_T, \qquad (n)$$

$M_T' = 0$, and the complete torque moment is taken by the side rod 2–2. By using the graphical solution (Fig. 65) within the limits determined by equations (m) and (n), and using equations (k) and (l) beyond these limits, the complete picture of the variation of the angle $\Delta\varphi$ can be obtained for $\pi/2 < \varphi < \pi$. In a similar manner other crank positions can be considered and a curve representing the variable spring characteristic $M_T/\Delta\varphi$ as a function of ωt, similar to that shown in Fig. 63, can be plotted.

Vibrations in the Side Rod Drive System.—Considering the motion of the system shown in Fig. 61, let

I_1 = moment of inertia of the mass rotating about the axis 0–0,

I_2 = moment of inertia of the mass rotating about the axis 0_1–0_1,

φ_1, φ_2 = corresponding angles of rotation about 0–0 and 0_1–0_1 respectively,

$\Delta\varphi = \varphi_1 - \varphi_2$ = the angular displacement of the motor with respect to the driving axle due to deformation in the shafts and side rods,

ψ = the variable flexibility of the side rod drive = the torque moment necessary to produce an angular displacement $\Delta\varphi$ equal to one radian.

In the particular case considered above (eq. 56, p. 94) we have

$$\psi = \frac{M_T}{\Delta\varphi} = \frac{a - b \cos 4\omega t}{c - d \cos 4\omega t}. \qquad (56)^1$$

M_T, M_r = moments of the external forces acting on the masses I_1 and I_2, respectively. When motion of the locomotive takes place a constant torque moment M_T acts on the mass having a moment of inertia I_1 (Fig. 61) and in opposition to this a moment $\psi(\varphi_1 - \varphi_2)$ is brought into play which represents the reaction of the elastic forces of the twisted shaft 0–0. The differential equation of motion will be

$$-I_1 \frac{d^2 \varphi_1}{dt^2} + M_T - \psi(\varphi_1 - \varphi_2) = 0. \qquad (a)$$

* The configuration shown in Fig. 62, in which the cranks are situated in the first and second quadrants is considered here.

In the same manner the differential equation of motion for the second mass will be

$$-I_2\frac{d^2\varphi_2}{dt^2} - M_r + \psi(\varphi_1 - \varphi_2) = 0. \tag{b}$$

In actual cases I_1 and I_2 represent usually the *equivalent* moments of inertia, the magnitude of which can be calculated from the consideration of the constitution of the system.

From equations (a) and (b) we have

$$\frac{d^2(\varphi_1 - \varphi_2)}{dt^2} + \psi\frac{I_1 + I_2}{I_1 I_2}(\varphi_1 - \varphi_2) = \frac{M_T}{I_1} + \frac{M_r}{I_2}.$$

Letting

$$\varphi_1 - \varphi_2 = x, \qquad \psi\frac{I_1 + I_2}{I_1 I_2} = \theta, \tag{57}$$

the following equation will be obtained:

$$\ddot{x} + \theta x = \frac{M_T}{I_1} + \frac{M_r}{I_2}, \tag{c}$$

in which θ is a certain periodical function of the time. In the case shown in Fig. 61, we have from eq. 56, p. 94.

$$\theta = \psi\frac{I_1 + I_2}{I_1 I_2} = \frac{I_1 + I_2}{I_1 I_2}\frac{a - b\cos 4\omega t}{c - d\cos 4\omega t}. \tag{57^1}$$

If the rigidity of the shaft is very large in comparison with that of the side rods, the quantities b and d in eq. (57)^1 can be neglected (see p. 94) and we obtain

$$\theta = K_2\frac{I_1 + I_2}{I_1 I_2}.$$

We arrive at a system having a constant flexibility, the period of free vibration of which can be easily found from the following equation (see p. 8)

$$\tau = 2\pi\sqrt{\frac{I_1 I_2}{K_2(I_1 + I_2)}}. \tag{58}$$

Under the action of a variable torque moment M_T large vibrations in the system may arise if the period of M_T is equal to or a multiple * of the period of the free vibrations of the system. In this manner a series of critical speeds for the system will be established.

* It is assumed that M_T is represented by a trigonometrical series (see paragraph 12); then resonance occurs if the period of one of the terms of this series becomes equal to τ.

In the case of a variable flexibility the problem becomes more complicated. Instead of definite *critical speeds*, there exist definite *regions of speeds* within which large vibrations may be built up. In order to determine the limits of these critical regions an investigation of the equation

$$\ddot{x} + \theta x = 0, \tag{59}$$

representing the free vibrations of the system becomes necessary. The factor θ in this equation is a periodic function of the time depending on the variable flexibility of the system and is determined by eq. (57). Let T denote the period of this function and $x(t)$ — a solution of eq. (59). Then, as was shown by E. Meissner,* the values of T corresponding to the limits of the *critical regions* are those values at which one of the two following conditions is fulfilled:

$$x(t + T) = x(t), \tag{d}$$

$$x(t + T) = -x(t). \tag{e}$$

In the further discussion we will call the case (*d*) eq. (59) a *periodic solution* of the first kind and the case (*e*) a *periodic solution* of the second kind. It means that values of T determining the limits of the *critical regions* are those values at which eq. (59) has periodic solutions either of the first or of the second kind.

Before giving an application of this general conclusion in calculating the critical regions, a more detailed discussion will be given of the differential eq. (59).

Discussion of the Differential Equation (59).†—The general solution of the linear differential eq. (59), which is of the second order can always be represented in the following form,

$$x = a_1\eta_1(t) + a_2\eta_2(t) \tag{60}$$

in which a_1 and a_2 are two arbitrary constants; $\eta_1(t)$ and $\eta_2(t)$ are two particular solutions of eq. (59), chosen so as to satisfy the following initial conditions:

$$\eta_1(0) = 1, \qquad \dot{\eta}_1(0) = 0, \qquad \eta_2(0) = 0, \qquad \dot{\eta}_2(0) = 1 \tag{a}$$

In the case of a constant flexibility, when $\theta = p^2$, we have for instance

$$\eta_1(t) = \cos pt; \qquad \eta_2(t) = \frac{\sin pt}{p}.$$

Assuming that the conditions (*a*) are fulfilled, the arbitrary constants in the general solution (60) have very simple meanings, namely

$$a_1 = (x)_{t=0} = x_0, \qquad a_2 = (\dot{x})_{t=0} = \dot{x}_0$$

* See the paper mentioned above, p. 91.

† The following analysis is only of theoretical interest and can be omitted without affecting the understanding of later sections.

This means that the complete displacement of the system during vibration given by the general solution (60) is composed of two parts, one—depending on the initial displacement x_o of the system and another depending on the initial velocity $\dot{x}_0$.

If an initial displacement and an initial velocity are given an approximate solution of eq. (59) can always be obtained by using the graphical or numerical methods described above in paragraphs 19 and 20.

In order to obtain a picture of the gradual building up of the vibrations, the integration must be extended over a period of time many times larger than the period T of the function θ. This requires considerable work and at the same time the accuracy of these calculations diminishes with the increase of t. It will now be shown that general conclusions about the character of solution (60) can be made assuming that the functions $\eta_1(t)$ and $\eta_2(t)$ are known within the limits $0 \leq t \leq T$. For obtaining these functions in the above interval it is only necessary to find solutions of eq. 60 for the two particular initial conditions expressed by

$$x_0 = 1; \qquad \dot{x}_0 = 0$$

and

$$x_0 = 0; \qquad \dot{x}_0 = 1.$$

This can be easily done by some approximate method and in this manner the functions $\eta_1(t)$ and $\eta_2(t)$ in the interval $0 \leq t \leq T$ will be obtained.

Due to the fact that the function θ in eq. (59) has the period T, this equation does not change if t will be replaced by $t + T$. Therefore $\eta_1(t + T)$ and $\eta_2(t + T)$ also are solutions of eq. (59) and must have the same form as solution (60), i.e.,

$$\begin{aligned} \eta_1(t + T) &= a\eta_1(t) + b\eta_2(t) \\ \eta_2(t + T) &= c\eta_1(t) + d\eta_2(t) \end{aligned} \tag{b}$$

Substituting $t = 0$ in these equations and in their derivatives with respect to time, the following values for the constants a, b, c, and d will be obtained,

$$\begin{aligned} a &= \eta_1(T), \qquad c = \eta_2(T) \\ b &= \dot{\eta}_1(T), \qquad d = \dot{\eta}_2(T) \end{aligned} \tag{c}$$

It is seen now that by knowing $\eta_1(t)$ and $\eta_2(t)$ in the interval $0 \leq t \leq T$ the values of these functions for the interval $T < t < 2T$ can be calculated by using the equations (b) and (c). Repeating the same reasoning, a further extension of the interval in which the values of the functions $\eta_1(t)$ and $\eta_2(t)$ will be known, can be obtained.

After these preliminary remarks we will now return again to eq. (59) and endeavor to obtain a solution $N(t)$ of this equation so that after an elapse of a time equal to the period T the vibration reproduces itself, but with enlarged amplitudes. This condition is represented as follows:

$$N(t + T) = kN(t) \tag{d}$$

As every solution of eq. (59) solution $N(t)$ has the form

$$N(t) = a_1\eta_1(t) + a_2\eta_2(t) \tag{e}$$

we have

$$N(t + T) = a_1\eta_1(t + T) + a_2\eta_2(t + T)$$

or, by using eq. (b)

$$N(t + T) = a_1\{a\eta_1(t) + b\eta_2(t)\} + a_2\{c\eta_1(t) + d\eta_2(t)\} \tag{f}$$

Now, substituting (*e*) and (*f*) in eq. (*d*), we have,

$$a_1\{a\eta_1(t) + b\eta_2(t)\} + a_2\{c\eta_1(t) + d\eta_2(t)\} = k\{a_1\eta_1(t) + a_2\eta_2(t)\}$$

This equation will be satisfied for every value of t if,

$$aa_1 + ca_2 = ka_1$$
$$ba_1 + da_2 = ka_2$$

From these equations we have

$$\frac{a_1}{a_2} = -\frac{c}{a-k} = -\frac{d-k}{b} \tag{g}$$

i.e., the factor k must satisfy the following equation,

$$(a-k)(d-k) = bc$$

or

$$k^2 - (a+d)k + ad - bc = 0 \tag{h}$$

This equation can be simplified if we take into consideration that $\eta_1(t)$ and $\eta_2(t)$ are solutions of eq. (59) so that

$$\ddot{\eta}_1(t) + \theta\eta_1(t) = 0$$
$$\ddot{\eta}_2(t) + \theta\eta_2(t) = 0$$

Eliminating from these equations θ, we have,

$$\ddot{\eta}_1(t)\eta_2(t) - \ddot{\eta}_2(t)\eta_1(t) = 0$$

from which, by integration, we obtain

$$\dot{\eta}_1(t)\eta_2(t) - \dot{\eta}_2(t)\eta_1(t) = \text{const.}$$

Substituting in this equation $t = 0$ and $t = T$ and using eq. (*a*) and eq. (*c*) we obtain

$$-1 = bc - ad$$

Now eq. (*h*) becomes

$$k^2 - (a+d)k + 1 = 0 \tag{k}$$

from which

$$k_{1,2} = \frac{a+d}{2} \pm \sqrt{\left(\frac{a+d}{2}\right)^2 - 1} \tag{l}$$

For each of these two roots the corresponding value of the ratio a_1/a_2 can be calculated from eq. (*g*) and in this manner two normal integrals (see eq. *e*) $N_1(t)$ and $N_2(t)$ will be obtained such that

$$N_1(t+T) = k_1N_1(t), \qquad N_2(t+T) = k_2N_2(t)$$

It is seen that the variation of the integrals $N_1(t)$ and $N_2(t)$ with the time will depend on the values of the roots k_1 and k_2 of the eq. (*k*).

If,

$$\left(\frac{a+d}{2}\right)^2 > 1 \tag{p}$$

both roots are real and due to the fact that $k_1k_2 = 1$, the absolute value of one of these roots will be larger than unity. This indicates that the corresponding integral increases indefinitely with the time. Take, for instance, $(k)_1 > 1$, then after an interval of time equal to nT, we have,

$$N_1(t+nT) = k_1^nN_1(t)$$

It is seen that the rate of increase of the amplitude of vibration will depend on the magnitude of $|k_1|$.

Condition (p) represents the critical regions in which heavy vibrations must be expected. If

$$\left(\frac{a+d}{2}\right)^2 < 1 \tag{q}$$

both roots of eq. (m) become imaginary and the corresponding vibrations always retain a finite amplitude.* The limits of the critical regions will be found from the condition

$$\left(\frac{a+d}{2}\right)^2 = 1$$

In this case,

$$k_1 = k_2 = +1$$

or

$$k_1 = k_2 = -1$$

and eq. (59) has either periodic solution of the first kind or periodic solutions of the second kind (see p. 99). If

$$k_1 = k_2 = 1,$$

we have

$$N(t+T) = N(t)$$

a *periodic solution* of the first kind while if,

$$k_1 = k_2 = -1,$$

we have,

$$N(t+T) = -N(t)$$

a *periodic solution* of the second kind (see p. 99).

In the first case the period can be represented by the formula $2T/n$, in which $n = 2, 4, 6, \cdots$. In the second case the period will be represented by the same formula but $n = 1, 3, 5, \cdots$. It is seen from this that for determining the limits of the critical regions for a system with variable flexibility it is only necessary to determine such values of the period T of the function θ in eq. (59) at which this equation has either a *periodic solution* of the first kind or a *periodic solution* of the second kind. These values of T will represent the limits of the critical regions.

Calculation of Regions of Critical Speeds.†

In the case of an electric locomotive, the fluctuation in the flexibility of the system is usually small and the regions of critical speed can be calculated by successive approximation. The procedure of these calculations will now be shown in a particular case where the function θ in the general equation,

$$\ddot{x} + \theta x = 0 \tag{59^1}$$

has the following form,

$$\theta = \frac{a + b\cos 2\omega t + c\cos 4\omega t}{p + q\cos 2\omega t + r\cos 4\omega t}\,\frac{I_1 + I_2}{I_1 I_2} \tag{61}$$

In the case of a symmetrical arrangement, discussed above, $b = q = 0$ and we arrive at the form shown in eq. (57)[1] p. 98.

* See the paper by Meissner, mentioned above, p. 91.

† See Karl E. Müller, Ueber die Schüttelschwingungen des Kuppelstangenantriebes," Dissertation der Eidgen. Techn. Hochschule in Zürich.

Assuming only small fluctuations in θ during motion of the locomotive, the quantities b, c, q and r in eq. (61) become small in comparison with a and p and, by performing the division, this equation can be represented in the following form,

$$\theta = \left\{ \frac{a}{p} + \frac{b}{p}\cos 2\omega t + \frac{c}{p}\cos 4\omega t \right\} \left\{ 1 - \left(\frac{q}{p}\cos 2\omega t + \frac{r}{p}\cos 4\omega t \right) + \left(\frac{q}{p}\cos 2\omega t + \frac{r}{p}\cos 4\omega t \right)^2 - \cdots \right\} \frac{I_1 + I_2}{I_1 I_2} \qquad (a)$$

Let,

$$\frac{a}{p}\frac{I_1 + I_2}{I_1 I_2} = g_o; \quad \frac{b}{p}\frac{I_1 + I_2}{I_1 I_2} = g_1\epsilon; \quad \frac{c}{p}\frac{I_1 + I_2}{I_1 I_2} = g_2\epsilon; \quad \frac{q}{p} = g_3\epsilon; \quad \frac{r}{p} = g_4\epsilon$$

where g_1, g_2, g_3 and g_4 are quantities of the same order as g_0 and ϵ denotes a small quantity. Then by using the identity,

$$2\cos 2mt \cos 2nt = \cos 2(m + n)t + \cos 2(m - n)t \qquad (b)$$

eq. (a) can be represented in the following form,

$$\theta = \pi^2\{a_0 + \epsilon(a_1 \cos 2\omega t + a_2 \cos 4\omega t) + \epsilon^2(a_3 \cos 2\omega t + a_4 \cos 4\omega t + a_5 \cos 6\omega t + a_6 \cos 8\omega t) + \epsilon^3(a_7 \cos 2\omega t + \cdots) + \cdots\} \qquad (c)$$

in which the constants a_0, a_1, a_2, $\cdots$ can be expressed by the quantities g_0, g_1, $\cdots$ given above.

It is seen now that the function θ, depending on the variable flexibility of the system, has a period

$$T = \frac{\pi}{\omega} \qquad (d)$$

i.e., two complete periods of θ correspond to one revolution of the crank.

In the following discussion of the differential equation (59)[1] the angle φ, a new independent variable, instead of the time t will be introduced. This variable will be determined by the equation,

$$\varphi = \omega t \qquad (e)$$

and will represent the angle of rotation of the crank,

$$\dot{x} = \frac{dx}{dt} = \omega\frac{dx}{d\varphi}, \qquad \ddot{x} = \frac{d^2x}{dt^2} = \omega^2\frac{d^2x}{d\varphi^2}$$

Substituting in eq. (59)[1] and using eq. (d), we have

$$\frac{d^2x}{d\varphi^2} + \frac{T^2}{\pi^2}\theta x = 0, \qquad (62)$$

in which, from eq. (c)

$$\theta = \pi^2\{a_0 + \epsilon(a_1 \cos 2\varphi + a_2 \cos 4\varphi) + \epsilon^2(a_3 \cos 2\varphi + a_4 \cos 4\varphi + a_5 \cos 6\varphi + a_6 \cos 8\varphi) + \epsilon^3(a_7 \cos 2\varphi + \cdots)\}, \qquad (f)$$

i.e., the period of function θ is now equal to π.

According to the previous discussion the limits of the *critical regions* of the motion of the system correspond to those values of the period T at which eq. (62) has *periodic* solutions of the first or second kind, i.e.,

$$x(\varphi + \pi) = x(\varphi) \qquad (g)$$

or

$$x(\varphi + \pi) = -x(\varphi). \tag{g$'$}$$

For calculating these particular values of T assume that T and $x(\varphi)$ are developed in the following series,

$$\begin{aligned} T &= \alpha_0 + \alpha_1\epsilon + \alpha_2\epsilon^2 + \alpha_3\epsilon^3 + \cdots, \\ x(\varphi) &= x_0(\varphi) + \epsilon x_1(\varphi) + \epsilon^2 x_2(\varphi) + \cdots, \end{aligned} \tag{h}$$

in which ϵ denotes the same small quantity as in eq. (f) above.

Substituting the series (f) and (h) in eq. (62) we have

$$\begin{aligned} \frac{d^2x_0(\varphi)}{d\varphi^2} + \epsilon\frac{d^2x_1(\varphi)}{d\varphi^2} + \epsilon^2\frac{d^2x_2(\varphi)}{d\varphi^2} + \cdots + (\alpha_0 + \alpha_1\epsilon + \alpha_2\epsilon^2 + \cdots)^2 \times \\ \times \{a_0 + \epsilon(a_1 \cos 2\varphi + a_2 \cos 4\varphi) + \epsilon^2(a_3 \cos 2\varphi + \cdots)\} \times \\ \times \{x_0(\varphi) + \epsilon x_1(\varphi) + \epsilon^2 x_2(\varphi) + \cdots\} = 0. \end{aligned}$$

Rearranging the left side of this equation in ascending powers of ϵ and equating the coefficients of every power of ϵ to zero, the following system of equations will be obtained

$$\frac{d^2x_0(\varphi)}{d\varphi^2} + \alpha_0^2 a_0 x_0(\varphi) = 0, \tag{k}$$

$$\frac{d^2x_1(\varphi)}{d\varphi^2} + \alpha_0^2 a_0 x_1(\varphi) + x_0(\varphi)\{2a_0\alpha_0\alpha_1 + \alpha_0^2(a_1 \cos 2\varphi + a_2 \cos 4\varphi)\} = 0. \tag{l}$$

. . . . . .

Equation (k) represents a simple harmonic motion, the solution of which can be written in the form:

$$x_0 = A \cos (n\varphi - \delta_0). \tag{m}$$

In which

$$n = \sqrt{a_0\alpha_0^2} \tag{n}$$

and A and δ_0 are arbitrary constants.

In order to satisfy the conditions (g) it is necessary to take $n = 2, 4, 6, \cdots$ for *periodic* solutions of the first kind and, $n = 1, 3, 5, \cdots$ for *periodic* solutions of the second kind. Substituting this in eq. (n) and taking into account that from eq. (h) α_0 is the first approximation for the period T, and $\pi^2 a_0 = \theta_0$ represents some average value of θ, we have,

$$T_n = \frac{n}{\sqrt{a_0}} = \frac{n\pi}{\sqrt{\theta_0}}, \tag{63}$$

in which $n = 1, 2, 3, \cdots$.

Comparing this result with $\tau = 2\pi/\sqrt{\theta_0}$ the period of the natural vibrations of a system having constant flexibility $\theta = \theta_0$, it can be concluded that as a first approximation definite critical speeds are obtained instead of *critical regions of speeds.* At this critical speed the period $2T$ of one revolution of the crank is equal to or a multiple of the period τ of the natural vibration of the system with a constant rigidity corresponding to some average value θ_0 of the function θ.

The second approximation for the solution of eq. (59)[1] will now be obtained by substituting the first approximation (m) in equation (l), This gives

$$\begin{aligned} \frac{d^2x_1(\varphi)}{d\varphi^2} + a_0\alpha_0^2 x_1(\varphi) = -2a_0\alpha_0\alpha_1 A \cos (n\varphi - \delta_0) \\ - \alpha_0^2 A \cos (n\varphi - \delta_0)(a_1 \cos 2\varphi + a_2 \cos 4\varphi), \end{aligned}$$

or by using eq. (*b*)

$$\frac{d^2x_1(\varphi)}{d\varphi^2} + a_0\alpha_0^2x_1(\varphi) = -2a_0\alpha_0\alpha_1A\cos(n\varphi - \delta_0) -$$

$$-\frac{\alpha_0^2a_1}{2}A\{\cos[(n+2)\varphi - \delta_0] + \cos[(n-2)\varphi - \delta_0]\} -$$

$$-\frac{\alpha_0^2a_2}{2}A\{\cos[(n+4)\varphi - \delta_0] + \cos[(n-4)\varphi - \delta_0]\}. \qquad (o)$$

The general solution of this equation consists of two parts: first the free vibration represented by

$$B\cos(n\varphi - \delta_1),$$

in which B and δ_1 represent two arbitrary constants while $n = \sqrt{a_0\alpha_0^2}$, and another part, a forced vibration. In calculating these latter vibrations the general expression (24), p. 20 will be used. Denoting by $R(\varphi)$ the right side of eq. (*o*) the forced vibration can be represented in the following form,

$$\frac{1}{n}\int_0^\varphi R(\xi)\sin n(\varphi - \xi)d\xi. \qquad (p)$$

The terms on the right side of eq. (*o*) have the general form

$$N\cos[(n \pm m)\varphi - \delta].$$

Substituting this in (*p*) we have

$$\frac{N}{n}\int_0^\varphi \cos[(n \pm m)\xi - \delta]\sin n(\varphi - \xi)d\xi = -\frac{N}{2n}\cdot\frac{1}{\pm m}\{\cos[(n \pm m)\varphi - \delta]$$

$$-\cos(n\varphi - \delta)\} + \frac{N}{2n}\frac{1}{2n \pm m}\{\cos[(n \pm m)\varphi - \delta] - \cos(n\varphi + \delta)\}. \qquad (q)$$

There are two exceptional cases $m = 0$ and $2n \pm m = 0$. In the first exceptional case ($m = 0$) the first term on the right side of eq. (*q*) becomes,

$$\frac{N}{2n}\varphi\sin(n\varphi - \delta).$$

In the second exceptional case $[(2n \pm m) = 0]$ the second term on the right side of equation (*q*) becomes

$$\frac{N}{2n}\varphi\sin(n\varphi + \delta).$$

After this preliminary discussion the general solution of eq. (*o*) can be represented in the following form,

$$x_1 = B\cos(n\varphi - \delta_1) - 2a_0\alpha_0\alpha_1A\frac{1}{2n}\varphi\sin(n\varphi - \delta_0)$$

$$-\frac{2a_0\alpha_0\alpha_1A}{(2n)^2}[\cos(n\varphi - \delta_0) - \cos(n\varphi + \delta_0)] -$$

$$-\frac{\alpha_0^2 a_1 A}{2}\begin{cases}\cos[(n+2)\varphi-\delta_0]\left(-\frac{1}{2n}\cdot\frac{1}{2}+\frac{1}{2n}\cdot\frac{1}{2n+2}\right)+\\ \cos[(n-2)\varphi-\delta_0]\left(\frac{1}{2n}\cdot\frac{1}{2}+\frac{1}{2n}\cdot\frac{1}{2n-2}\right)+\\ \cos(n\varphi-\delta_0)\left(\frac{1}{2n}\cdot\frac{1}{2}-\frac{1}{2n}\cdot\frac{1}{2}\right)+\\ \cos(n\varphi+\delta_0)\left(-\frac{1}{2n}\cdot\frac{1}{2n+2}-\frac{1}{2n}\cdot\frac{1}{2n-2}\right),\end{cases} \tag{r}$$

$$-\frac{\alpha_0^2 a_2 A}{2\cdot 2n}\begin{cases}\cos[(n+4)\varphi-\delta_0]\left(-\frac{1}{4}+\frac{1}{2n+4}\right)+\\ \cos[(n-4)\varphi-\delta_0]\left(\frac{1}{4}+\frac{1}{2n-4}\right)+\\ \cos(n\varphi-\delta_0)\left(\frac{1}{4}-\frac{1}{4}\right)+\\ \cos(n\varphi+\delta_0)\left(\frac{1}{2n+4}-\frac{1}{2n-4}\right).\end{cases}$$

It is seen that all the terms of the obtained solution, except the term,

$$-2a_0\alpha_0\alpha_1 A\frac{1}{2n}\varphi\sin(n\varphi-\delta),$$

are periodical of the first or second kind; therefore the conditions (g) will be satisfied by putting,

$$\alpha_1 = 0.$$

In this manner the second approximation for T will be obtained from the first of the equations (h) which approximation coincides with the first approximation. Exceptional cases only occur if $n = 1$ and $n = 2$.

In the case $n = 1$, the terms

$$-\frac{\alpha_0^2 a_1 A}{2}\left\{\cos[(n-2)\varphi-\delta_0]\frac{1}{2n}\cdot\frac{1}{2n-2}-\cos(n\varphi+\delta_0)\frac{1}{2n}\frac{1}{2n-2}\right\}$$

in the general solution (r) assume the form $\infty - \infty$ and must be replaced by the term

$$-\frac{\alpha_0^2 a_1 A}{2}\cdot\frac{1}{2}\cdot\varphi\sin(\varphi+\delta_0).$$

In order to make solution (r) periodical of the first or second kind it is necessary to put in this case

$$-2a_0\alpha_0\alpha_1 A\frac{1}{2}\varphi\sin(\varphi-\delta_0)-\frac{\alpha_0^2 a_1 A}{2}\frac{1}{2}\varphi\sin(\varphi+\delta_0)=0$$

or

$$\sin\varphi\cos\delta_0\left(-a_0\alpha_0\alpha_1-\frac{\alpha_0^2 a_1}{4}\right)+\cos\varphi\sin\delta_0\left(a_0\alpha_0\alpha_1-\frac{\alpha_0^2 a_1}{4}\right)=0. \tag{s}$$

There are two possibilities to satisfy this equation:

(1) $$\delta_0 = 0, \qquad -a_0\alpha_0\alpha_1-\frac{\alpha_0^2 a_1}{4}=0, \qquad \alpha_1=-\frac{\alpha_0 a_1}{4a_0}$$

or

(2) $$\delta_0 = \frac{\pi}{2}, \qquad a_0\alpha_0\alpha_1-\frac{\alpha_0^2 a_1}{4}=0, \qquad \alpha_1=\frac{\alpha_0 a_1}{4a_0}.$$

Substituting the obtained values of α_1 in the first of the eqs. (h) and taking into account that, from eq. (n) for $n = 1$, $\alpha_0 = \frac{1}{\sqrt{a_0}}$, we have as the second approximation for T_1, corresponding to the boundaries of the critical region,

$$T_{1\min} = \frac{1}{\sqrt{a_0}} - \epsilon \frac{1}{\sqrt{a_0}} \cdot \frac{a_1}{4a_0},$$
$$T_{1\max} = \frac{1}{\sqrt{a_0}} + \epsilon \frac{1}{\sqrt{a_0}} \frac{a_1}{4a_0}. \tag{64}$$

It is seen that instead of a critical speed, given for $n = 1$ by eq. 63, we obtain a *critical region* between the limits $(T_1)_{\min}$ and $(T_1)_{\max}$. The extension of this region depends on the magnitude of the small quantity ϵ and it diminishes with the diminishing of the fluctuation in the flexibility of the system. It is interesting to note also that the difference in phase δ_0 between the function θ and the free vibration of the system has two definite values for the two limiting conditions, $\delta_0 = 0$ and $\delta_0 = \pi/2$. It should be noted also that the critical region considered above ($n = 1$) corresponds to the highest speed of rotation and is practically the most dangerous region.

For the case $n = 2$, i.e., for the next lower critical region, by using the same method as above, we will obtain,

$$(T_2)_{\substack{\min \\ \max}} = \sqrt{\frac{4}{a_0}} \mp \epsilon \sqrt{\frac{4}{a_0}} \frac{a_2}{4a_0}. \tag{65}$$

In order to obtain the critical regions corresponding to $n = 3$ and $n = 4$ the third approximation and the equation for x_2 must be considered. This equation can be obtained from the general eq. (o) in the same manner as the equations (k) and (l), used above for calculating the first and the second approximations.

By using the described method the critical regions for the equation (59) representing free vibrations in a locomotive can be established. These regions are exactly those in which heavy vibrations under the action of external forces (see eq. c, p. 98) may occur.* The investigation of actual cases shows † that the extensions of the critical regions are small and that the first approximation in which the variable rigidity is replaced by some average constant rigidity and in which the critical speeds are given by eq. 63, gives a good approximation to the actual distribution of critical speeds.

In our investigation only displacements due to elastic deformations in the system were considered. In actual conditions the problem of locomotive vibrations is much more complicated due to various kinds of clearances which always are present in the actual structure and the effect of which on the flexibility of the system have already been discussed. When the speed of a moving locomotive attains a critical region, heavy vibrations of the system may begin in which the moving masses will cross the clearances twice during every cycle.‡ The conditions will then become analogous to those shown in Fig. 44, p. 70. Such a kind of motion is accompanied with impact

* See the paper by Prof. E. Meissner, mentioned above, p. 91.

† See the paper by K. E. Müller, mentioned above, p. 102.

‡ The possibility of the occurrence of this type of vibration can be removed to a great extent by introducing a flexible gear system.

and is very detrimental in service. Many troublesome cases, especially in the earlier period of the building of electric locomotives, must be attributed to these vibrations. For excluding this type of vibrations a flexibility of the system must be so chosen that the operating speed of the locomotive is removed as far as possible from the critical regions. Experience shows that the detrimental effect of these vibrations can be minimized by the introduction into the system of flexible members such as, for instance, flexible gears. In this manner the fluctuation in flexibility of the system will be reduced and the extension of the critical regions of speed will be diminished. The introduction in the system of an additional damping can also be useful because it will remove the possibility of a progressive increase in the amplitude of vibrations.*

* Various methods of damping are discussed in the book by A. Wichert mentioned above, p. 92.

CHAPTER III

SYSTEMS HAVING SEVERAL DEGREES OF FREEDOM

24. d'Alembert's Principle and the Principle of Virtual Displacements. In the previous discussion of the vibration of systems having one degree of freedom d'Alembert's principle has often been used (see paragraph 1, p. 2). The same principle can also be applied to systems with several degrees of freedom.

As a first example the motion of a particle free in space will be considered. For determining the position of this particle three coordinates are necessary. By taking Cartesian coordinates and denoting by X, Y and Z the components of the resultant of all the forces acting on the point, the equations of equilibrium of the point will be,

$$X = 0, \qquad Y = 0, \qquad Z = 0. \tag{66}$$

If the point is in motion the differential equations of motion can be written in the same manner as the equations of statics, using d'Alembert's principle. It is only necessary to add the inertia force to the given external forces. The components of this force in the x, y and z directions are $-m\ddot{x}$, $-m\ddot{y}$, $-m\ddot{z}$, respectively, and the equations of motion will be

$$X - m\ddot{x} = 0, \qquad Y - m\ddot{y} = 0, \qquad Z - m\ddot{z} = 0. \tag{67}$$

If a system of several particles free in space is considered, the equations (67) should be written for every point of the system.

Consider now systems in which the displacements of the particles constituting the system are not entirely independent but are subjected to certain constraints, which can be expressed in the form of equations between the coordinates of these points. In Fig. 66, several simple cases of such systems are represented. In the case of a spherical pendulum (Fig. 66, a) the distance of the particle m from the origin O should remain constant during motion and equal to the length l of the pendulum. Therefore, the coordinates x, y and z of this point are no longer independent: they have to satisfy the equation *

$$x^2 + y^2 + z^2 = l^2. \tag{a}$$

* The cases when the equations of constraint include not only the coordinates but also the velocities of the particles and the time will not be considered here. We do not consider, for instance, the oscillation of a pendulum, the length of which has to vary during the motion by a special device, i.e., the case where l is a certain function of t.

In the case of a double pendulum (Fig. 66, *b*) the conditions of constraint will be represented by the following equations,

$$x_1^2 + y_1^2 = l_1^2, \tag{b}$$

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = l_2^2. \tag{c}$$

In the case of a connecting rod system (Fig. 66, *c*) the point A is moving along a circle of radius r and the point B is moving along the x

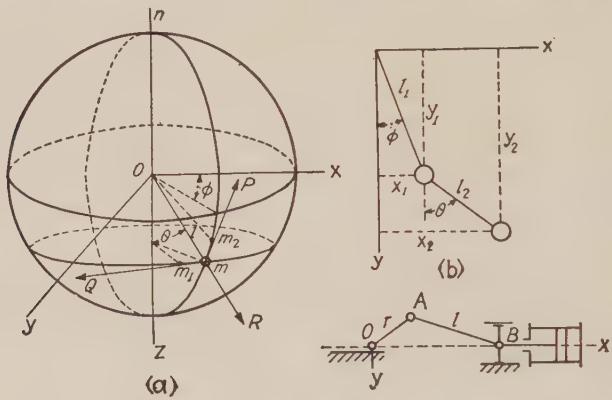


Fig. 66.

axis. The position of the system can be specified by one coordinate only, i.e., the system has only one degree of freedom.

In considering the conditions of equilibrium of such systems the *principle of virtual displacements* will be applied. This principle states that, if a system is in equilibrium, the work done by the forces on every *virtual displacement* (small possible displacement i.e., displacement which can be performed without violating the constraints of the system) of the system must be equal to zero. For instance, in the case of a spherical pendulum denoting by X, Y and Z, the components of the resultant of all the forces acting on the mass m, the equation of virtual displacements will be

$$X\delta x + Y\delta y + Z\delta z = 0, \tag{d}$$

in which δx, δy, δz, are the components of the virtual displacement of the point m, i.e., small changes of the coordinates x, y, z of m satisfying the condition of constraint (*a*). Then,

$$(x + \delta x)^2 + (y + \delta y)^2 + (z + \delta z)^2 = l^2$$

or neglecting small quantities of higher order,

$$x\delta x + y\delta y + z\delta z = 0.$$

This equation shows that a virtual displacement is perpendicular to the bar l of the pendulum and that any small displacement of the point m on the surface of the sphere can be considered as a virtual displacement. Equation (d) will be satisfied if the resultant of all forces acting on m be normal to the spherical surface, because only in such a case the work done by these forces on every virtual displacement will be equal to zero.

Combining now the principle of virtual displacements with d'Alembert's principle the differential equations of motion of a system with constraints can be easily obtained. For instance, in the case of a spherical pendulum, by adding the inertia force to the external forces acting on the particle m, the following general equation of motion will be obtained,

$$(X - m\ddot{x})\delta x + (Y - m\ddot{y})\delta y + (Z - m\ddot{z})\delta z = 0, \tag{68}$$

in which δx, δy, δz are components of a virtual displacement, i.e., small displacement satisfying the condition of constraint (a). In the same manner for a system consisting of n particles m_1, m_2, m_3, $\cdots$ and subjected to the action of the forces X_1, Y_1, Z_1, X_2, Y_2, Z_2, $\cdots$ the general equation of motion

$$\sum_{i=1}^{i=n}\left\{(X_i - m_i\ddot{x}_i)\delta x_i + (Y_i - m_i\ddot{y}_i)\delta y_i + (Z_i - m_i\ddot{z}_i)\delta z_i\right\} = 0 \tag{69}$$

will be obtained, in which δx_i, δy_i, δz_i, $\cdots$ are components of virtual displacement, i.e., small displacements satisfying the conditions of constraint of the system. For instance, in the case of a double pendulum (Fig. (b)) the virtual displacements should satisfy the conditions (see eq. b, c)

$$(x_1 + \delta x_1)^2 + (y_1 + \delta y_1)^2 = l_1^2,$$
$$(x_2 + \delta x_2 - x_1 - \delta x_1)^2 + (y_2 + \delta y_2 - y_1 - \delta y_1)^2 = l_2^2.$$

It should be noted that X_i, Y_i and Z_i denote the components of the resultant of all the forces acting on the particle m_i, but there are several kinds of forces which do not do work on the virtual displacements; among these for instance are the reactions of connecting rods of invariable length, the reactions of fixed pins, the reactions of smooth surfaces or curves with which the moving particles are constrained to remain in contact. In the following it is assumed that in X_i, Y_i and Z_i only forces which produce work on the virtual displacements are included.

If there are no constraints and the particles of the system are com-

pletely free, the small quantities δx_i, δy_i, δz_i in eq. (69) are entirely independent and equation (69) will be satisfied for every value of the virtual displacements only in the case that for every particle of the system the equations

$$X_i - m\ddot{x}_i = 0, \qquad Y_i - m\ddot{y}_i = 0, \qquad Z_i - m\ddot{z}_i = 0$$

are satisfied. These are the equations (67) previously obtained for the motion of a free particle.

Equation (69) is a general equation of motion for a system of particles from which the necessary number of equations of motion, equal to the number of degrees of freedom of the system, can be obtained. The derivation of these equations will be shown in paragraph 26.

25. Generalized Coordinates and Generalized Forces.—In the previous paragraph where Cartesian coordinates were used it was shown that these coordinates, as applied for describing the configuration of a system, are usually not independent. Moreover, they must satisfy certain equations of constraint, for instance, the equations (*a*), (*b*) and (*c*) depending on the arrangement of the system. It is usually more convenient to describe the configuration of a system by means of quantities which are completely independent of each other. It is not necessary that these quantities have the dimension of a length. They may have other dimensions; for instance, it is sometimes useful to take for coordinates the angles between certain directions or the magnitudes of certain areas or certain volumes. Such independent quantities chosen for describing the configuration of a system are usually called *generalized coordinates.**

Take, for instance, the previously discussed case of a spherical pendulum (Fig. 66, *a*). The position of the pendulum will be completely determined by the two angles φ and θ shown in the figure. These two independent quantities can be taken as generalized coordinates for this case. The Cartesian coordinates of the point m can easily be expressed by the new coordinates φ and θ. Projecting the length of the pendulum Om on the coordinate axes, we have

$$\begin{aligned} x &= l \sin\theta \cos\varphi, \\ y &= l \sin\theta \sin\varphi, \\ z &= l \cos\theta. \end{aligned} \qquad (a)$$

In the case of a double pendulum (Fig. 66, *b*) the angles φ and θ,

* The terminology of "generalized coordinates," "velocities," "forces" was introduced by Thomson and Tait, Natural Philosophy, 1st edition, Oxford, 1867.

shown in the figure, can be used as generalized coordinates and the Cartesian coordinates will be expressed by these new coordinates as follows:

$$x_1 = l_1 \sin \varphi, \qquad y_1 = l_1 \cos \varphi, \tag{b}$$
$$x_2 = l_1 \sin \varphi + l_2 \sin \theta, \qquad y_2 = l_1 \cos \varphi + l_2 \cos \theta.$$

If a solid body of a homogeneous and isotropic material be subjected to a uniform external pressure p all its dimensions will be diminished in the same proportion and the new configuration will be completely determined by the change v of the volume V of the body. The quantity v can be taken as the generalized coordinate for this case.

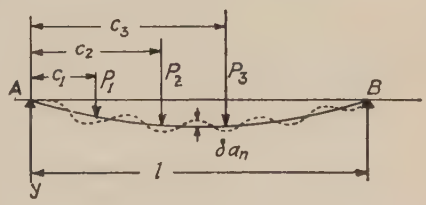


Fig. 67.

Consider now the bending of a beam supported at the ends (Fig. 67). In order to describe the configuration of this elastic system an infinitely large number of coordinates is necessary. The deflection curve can be defined by giving the deflection at every point of the beam, or we can proceed otherwise and represent the deflection curve in the form of a trigonometrical series:

$$y = a_1 \sin \frac{\pi x}{l} + a_2 \sin \frac{2\pi x}{l} + a_3 \sin \frac{3\pi x}{l} + \cdots. \tag{c}$$

The deflection curve will be completely determined if the coefficients $a_1, a_2, a_3, \cdots$ are given. These quantities may be taken for generalized coordinates in the case of the bending of a beam with supported ends.

If generalized coordinates are used for describing the configuration of a system, all the independent types of virtual displacements of the system can be obtained by giving small increases consecutively to everyone of these coordinates. By giving, for instance, a small increase $\delta\varphi$ to the angle φ in the case of the spherical pendulum a small displacement $mm_1 = l \sin \theta \delta\varphi$ along the parallel circle will be obtained. An increase of the coordinate θ by a small quantity $\delta\theta$ corresponds to a small displacement $mm_2 = l\delta\theta$ in the meridianal direction. Any other small displacement of the point m can always be resolved into two components such as mm_1 and mm_2.

In the case of bending of a beam with supported ends (Fig. 67) a small increase δa_n of any generalized coordinate a_n (see eq. c) involves a virtual deflection $\delta a_n \sin (n\pi x/l)$ represented in the figure by the dotted

line and having n half waves. Any departure of the beam from the position of equilibrium can be obtained by superimposing such sinusoidal displacements.

Applying generalized coordinates in our discussion we arrive at the notion of *generalized force.* There is a certain relation between a generalized coordinate and the corresponding generalized force, which we will explain first on simple examples. Returning again to the case of the spherical pendulum let P, Q and R represent the components of a force acting on the particle in the directions of the tangents to the meridian and to the parallel circle and in radial direction, respectively. If a small increase $\delta\varphi$ be given to the coordinate φ the point m will perform a small displacement $mm_1 = l \sin \theta \delta\varphi$ and the force acting on this point will do work equal to

$$Q\overline{mm_1} = Ql \sin \theta \delta\varphi.$$

The factor $Ql \sin \theta$, which must be multiplied by the increase $\delta\varphi$ of the generalized coordinate φ in order to obtain the work done during the displacement $\delta\varphi$, is called the *generalized force* corresponding to the coordinate φ. In this manner we get the complete analogy with the expression $X\delta x$ of the work of the force X on the displacement δx, in the direction of the force. In the case under consideration this "force" has a simple physical meaning. It represents the moment of the forces acting on the point m about the vertical diameter n–n. In the same manner it can be shown that the generalized force corresponding to the coordinate θ of the spherical pendulum will be represented by the moment of forces acting on the point m about the diameter perpendicular to the plane mon.

In the case of a body subjected to the action of a uniform hydrostatic pressure p by taking the decrease of the volume v as the generalized coordinate the corresponding generalized force will be the pressure p, because the quantity pv represents the work done by the external forces during the "displacement" v.

Let us consider now a more complicated case, namely, a beam under the action of the bending forces P_1, P_2, $\cdots$ (see Fig. 67). By taking the generalized expression (c) for the deflection curve and considering a_1, a_2, a_3, $\cdots$ as the generalized coordinates, the generalized force corresponding to one of these coordinates, such as a_n, will be found from a consideration of the work done by all the forces on the displacement δa_n. This displacement is represented in the figure by the dotted line.

In calculating the work produced during this displacement not only the external loads P_1, P_2 and P_3 but also internal forces of elasticity of

the beam must be taken into consideration. The vertical displacements of the points of application of the loads P_1, P_2, P_3, corresponding to the increase δa_n of the coordinate a_n, will be $\delta a_n \sin (n\pi c_1/l)$, $\delta a_n \sin (n\pi c_2/l)$ and $\delta a_n \sin (n\pi c_3/l)$, respectively. The work done by P_1, P_2 and P_3 during this displacement is

$$\delta a_n \left(P_1 \sin \frac{n\pi c_1}{l} + P_2 \sin \frac{n\pi c_2}{l} + P_3 \sin \frac{n\pi c_3}{l} \right). \tag{d}$$

In order to find out the work done by the forces of elasticity the expression for the potential energy of bending will be used. In the case of a beam of uniform cross section this energy is

$$V = \frac{EI}{2} \int_0^l \left(\frac{d^2y}{dx^2} \right)^2 dx, \tag{e}$$

in which EI denotes the *flexural rigidity* of the beam.

Substituting in this equation the series (c) for y and taking into consideration that

$$\int_0^l \sin \frac{m\pi x}{l} \sin \frac{n\pi x}{l} dx = 0; \quad \int_0^l \sin^2 \frac{m\pi x}{l} dx = \frac{l}{2},$$

where m and n denote different integer numbers, we obtain,

$$V = \frac{EI}{2} \left(\frac{a_1^2\pi^4}{2l^3} + \frac{a_2^2 2^4 \pi^4}{2l^3} + \cdots \right) = \frac{EI\pi^4}{4l^3} \sum_{n=1}^{n=\infty} a_n^2 n^4. \tag{f}$$

The increase of the potential energy of bending due to the increase δa_n of the coordinate a_n will be from eq. (f)

$$\frac{\partial V}{\partial a_n} \delta a_n = \frac{EI\pi^4}{2l^3} n^4 a_n \delta a_n. \tag{g}$$

This increase in potential energy is due to the work of the forces of elasticity. The work done by these forces is equal to (g) but with the opposite sign. Now, from (d) and (g) the generalized force corresponding to the coordinate a_n of the system shown in Fig. 67 will be

$$P_1 \sin \frac{n\pi c_1}{l} + P_2 \sin \frac{n\pi c_2}{l} + P_3 \sin \frac{n\pi c_3}{l} - \frac{EI\pi^4}{2l^3} n^4 a_n. \tag{h}$$

Proceeding in the same manner the generalized forces can be found also in any other case. Denoting in the general case by q_1, q_2, q_3, $\cdots$ the generalized coordinates of a system, the corresponding generalized forces Q_1, Q_2, Q_3, $\cdots$ will be found from the conditions that $Q_1\delta q_1$ represents

the work produced by all the forces during the displacement δq_1; in the same manner $Q_2\delta q_2$ represents the work done during the displacement δq_2 and so on.

26. Lagrange's Equations.—In deriving the general equation of motion (69) by using d'Alembert's principle it was pointed out that the components δx_i, δy_i, δz_i of the virtual displacements are not independent of each other and that they must satisfy certain conditions of constraint depending on the particular arrangement of the system. A great simplification in the derivation of the equations of motion of a system may be obtained by using independent generalized coordinates and generalized forces. Let $q_1, q_2, q_3, \cdots q_n$ be the generalized coordinates of a system with n degrees of freedom and let equations such as

$$x_i = \varphi_i(q_1, q_2 \cdots q_n); \qquad y_i = \psi_i(q_1, q_2 \cdots q_n); \qquad z_i = \theta_i(q_1, q_2 \cdots q_n) \quad (a)$$

represent the relations between the Cartesian and the generalized coordinates. It is assumed that these equations do not contain explicitly the time t and the velocities $\dot{q}_1, \dot{q}_2, \cdots \dot{q}_n$.

In order to transform the general equation (69) to these new coordinates let us write it down in the following form,

$$\sum_{i=1}^{i=n} m_i(\ddot{x}_i\delta x_i + \ddot{y}_i\delta y_i + \ddot{z}_i\delta z_i) = \sum_{i=1}^{i=n} X_i\delta x_i + Y_i\delta y_i + Z_i\delta z_i \qquad (b)$$

and consider a virtual displacement corresponding to an increase δq_i of some one generalized coordinate q_i only. Then it follows at once from the definition of generalized coordinate and generalized force (see paragraph 25) that the right side of eq. (b) representing the work done on the virtual displacement, is equal to

$$Q_i\delta q_i, \qquad (c)$$

where Q_i represents the generalized force corresponding to the coordinate q_i.

In order to transform the left side of eq. (b) to the new coordinates it should be noted that in the case under consideration when the coordinate q_i alone varies, the changes of the coordinates x_i, y_i, z_i will be

$$\delta x_i = \frac{\partial x_i}{\partial q_i}\delta q_i; \qquad \delta y_i = \frac{\partial y_i}{\partial q_i}\delta q_i; \qquad \delta z_i = \frac{\partial z_i}{\partial q_i}\delta q_i,$$

in which the symbol $\partial/\partial q_i$ denotes the partial derivative with respect to q_i, and x_i, y_i, z_i are given by eq. (a).

Then,

$$\sum_{i=1}^{i=n} m_i(\ddot{x}_i\delta x_i + \ddot{y}_i\delta y_i + \ddot{z}_i\delta z_i)$$

$$= \sum_{i=1}^{i=n} m_i\left(\ddot{x}_i\frac{\partial x_i}{\partial q_i} + \ddot{y}_i\frac{\partial y_i}{\partial q_i} + \ddot{z}_i\frac{\partial z_i}{\partial q_i}\right)\delta q_i$$

$$= \frac{d}{dt}\sum_{i=1}^{i=n} m_i\left(\dot{x}_i\frac{\partial x_i}{\partial q_i} + \dot{y}_i\frac{\partial y_i}{\partial q_i} + \dot{z}_i\frac{\partial z_i}{\partial q_i}\right)\delta q_i - \tag{d}$$

$$- \sum_{i=1}^{i=n} m_i\left(\dot{x}_i\frac{d}{dt}\frac{\partial x_i}{\partial q_i} + \dot{y}_i\frac{d}{dt}\frac{\partial y_i}{\partial q_i} + \dot{z}_i\frac{d}{dt}\frac{\partial z_i}{\partial q_i}\right)\delta q_i.$$

This equation can be simplified by using the expression

$$T = 1/2\sum_{i=1}^{i=n} m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)$$

for the kinetic energy of the system.

Remembering the equations (a) from which the velocities $\dot{x}_i$, $\dot{y}_i$, $\dot{z}_i$ can be represented as functions of the generalized coordinates q_i and the generalized velocities $\dot{q}_i$, we obtain the following expressions for the partial derivatives $\partial T/\partial\dot{q}_i$ and $\partial T/\partial q_i$

$$\frac{\partial T}{\partial\dot{q}_i} = \sum_{i=1}^{i=n} m_i\left(\dot{x}_i\frac{\partial\dot{x}_i}{\partial\dot{q}_i} + \dot{y}_i\frac{\partial\dot{y}_i}{\partial\dot{q}_i} + \dot{z}_i\frac{\partial\dot{z}_i}{\partial\dot{q}_i}\right), \tag{e}$$

$$\frac{\partial T}{\partial q_i} = \sum_{i=1}^{i=n} m_i\left(\dot{x}_i\frac{\partial\dot{x}_i}{\partial q_i} + \dot{y}_i\frac{\partial\dot{y}_i}{\partial q_i} + \dot{z}_i\frac{\partial\dot{z}_i}{\partial q_i}\right). \tag{f}$$

Taking into consideration that

$$\dot{x} = \frac{dx}{dt} = \frac{\partial x}{\partial q_1}\dot{q}_1 + \frac{\partial x}{\partial q_2}\dot{q}_2 + \cdots\frac{\partial x}{\partial q_n}\dot{q}_n, \tag{g}$$

we have

$$\frac{\partial\dot{x}}{\partial\dot{q}_1} = \frac{\partial x}{\partial q_1}; \qquad \frac{\partial\dot{x}}{\partial\dot{q}_2} = \frac{\partial x}{\partial q_2}; \qquad \frac{\partial\dot{x}}{\partial\dot{q}_n} = \frac{\partial x}{\partial q_n}.$$

Now eq. (e) can be written as follows

$$\frac{\partial T}{\partial\dot{q}_i} = \sum_{i=1}^{i=n} m_i\left(\dot{x}_i\frac{\partial x_i}{\partial q_i} + \dot{y}_i\frac{\partial y_i}{\partial q_i} + \dot{z}_i\frac{\partial z_i}{\partial q_i}\right). \tag{h}$$

In considering eq. (f) we note that

$$\frac{d}{dt}\frac{\partial x_i}{\partial q_i} = \frac{\partial^2 x_i}{\partial q_1\partial q_i}\dot{q}_1 + \frac{\partial^2 x_i}{\partial q_2\partial q_i}\dot{q}_2 + \cdots + \frac{\partial^2 x_i}{\partial q_n\partial q_i}\dot{q}_n,$$

or by using (g)

$$\frac{d}{dt}\frac{\partial x_i}{\partial q_i} = \frac{\partial}{\partial q_i}\frac{dx_i}{dt} = \frac{\partial \dot{x}}{\partial q_i}.$$

In the same manner we have,

$$\frac{d}{dt}\frac{\partial y_i}{\partial q_i} = \frac{\partial \dot{y}}{\partial q_i}, \qquad \frac{d}{dt}\frac{\partial z_i}{\partial q_i} = \frac{\partial \dot{z}_i}{\partial q_i}.$$

Substituting this into eq. (f) we obtain

$$\frac{\partial T}{\partial q_i} = \sum_{i=1}^{i=n} m_i\left(\dot{x}_i\frac{d}{dt}\frac{\partial x_i}{\partial q_i} + \dot{y}_i\frac{d}{dt}\frac{\partial y_i}{\partial q_i} + \dot{z}_i\frac{d}{dt}\frac{\partial z_i}{\partial q_i}\right). \tag{k}$$

Now by using eq. (h) and (k) the expression (d) representing the left side of eq. (b) can be written as follows

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i}.$$

By using for the right side of the same equation the expression (c) we finally obtain,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i. \tag{70}$$

This is the Lagrangian form * of the differential equations of motion. Such an equation can be written down for every generalized coordinate of the system so that finally the number of equations will be equal to the number of generalized coordinates, that is, the number of degrees of freedom of the system.

So far the generalized forces Q_1, Q_2, $\cdots$ have not been subject to any restriction. They may be constant forces or functions of either time, position or velocity. Consider now the particular case of forces having a potential and let V denote the potential energy of the system. Then from the condition that the work done on a virtual displacement is equal to the decrease in potential energy we have

$$Q_1\delta q_1 + Q_2\delta q_2 + Q_3\delta q_3 + \cdots = -\frac{\partial V}{\partial q_1}\delta q_1 - \frac{\partial V}{\partial q_2}\delta q_2 - \frac{\partial V}{\partial q_3}\delta q_3 - \cdots,$$

or by taking into account that the small displacements δq_1, δq_2, $\cdots$ are independent we obtain,

$$Q_1 = -\frac{\partial V}{\partial q_1}; \qquad Q_2 = -\frac{\partial V}{\partial q_2}; \qquad Q_3 = -\frac{\partial V}{\partial q_3}; \cdots$$

* Lagrange, Mécanique Analytique, Paris, 1788.

and the Lagrangian equation (70) takes the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0. \tag{71}$$

If there are acting on the system two kinds of forces: (1) forces having a potential and (2) other forces, for which we will retain the previous notations Q_1, Q_2, $Q_3 \cdots$ Lagrange's equations become

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i. \tag{72}$$

It was assumed in our previous discussion that the equations (a) representing the geometrical relations between the Cartesian and the generalized coordinates do not contain the time t explicitly. By using the same method as above it can be shown, however, that Lagrange's equations retain their form also in the case when the equations (a) vary continuously with the time, being of the type

$$x_i = \varphi_i(t, q_1, q_2 \cdots q_n); \quad y_i = \psi_i(t, q_1, q_2 \cdots q_n); \quad z_i = \theta_i(t, q_1, q_2 \cdots q_n).$$

An example of such a system will be obtained, if, for instance, we assume that the length l of a spherical pendulum shown in Fig. 66 does not remain constant but by some special arrangement is continuously varied with time.

27. The Spherical Pendulum.—As a simple example of the application of Lagrange's equation to the solution of dynamical problems the case of the spherical pendulum (Fig. 66, a) will now be considered. By using the angles φ and θ as generalized coordinates of the particle m the velocity of m will be,

$$v = \sqrt{(l\dot{\theta})^2 + (l \sin \theta \dot{\varphi})^2}$$

and the kinetic energy of the system is

$$T = \frac{m}{2}\{(l\dot{\theta})^2 + (l \sin \theta \dot{\varphi})^2\}. \tag{a}$$

Assuming that the weight mg is the only force acting on the mass m and proceeding as explained in paragraph 25, we find that the generalized force corresponding to the coordinate φ is equal to zero and the force, corresponding to the coordinate θ becomes

$$\Theta = -mgl \sin \theta. \tag{b}$$

Using (a) and (b) the following two equations of motion will be obtained

from the general equation (70)

$$\ddot{\theta} - \cos\theta \sin\theta \dot{\varphi}^2 = -\frac{g \sin\theta}{l}, \tag{c}$$

$$\sin^2\theta \dot{\varphi} = \text{const} = h, \text{ say.} \tag{d}$$

Several particular cases of motion will now be considered.

If the initial velocity of the particle m is in the direction of a tangent to a meridian, the path of the point will coincide with this meridian, i.e., $\dot{\varphi} = 0$, and the equation (c) above reduces to the known equation,

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

for the simple pendulum.

The case of a conical pendulum will be obtained by assuming that the angle θ remains constant during motion, then $\dot{\varphi}$ must also be a constant, according to eq. (d).

Let

$$\theta = \alpha; \qquad \dot{\varphi} = \omega,$$

Then, from eq. (c), and (d)

$$\omega^2 = \frac{g}{l\cos\alpha} = \frac{h^2}{\sin^4\alpha}, \tag{e}$$

from which the angular velocity ω and the constant h corresponding to a given angle α of a conical pendulum can be calculated.

Consider now a more complicated case where the steady motion of the mass m of the conical pendulum along a horizontal circle is slightly disturbed so that small oscillations of this mass about the horizontal circle take place. Let,

$$\theta = \alpha + \xi, \tag{f}$$

where ξ denotes the small fluctuation in the angle θ during this motion. Retaining in all further calculations only the first power of the small quantity ξ we obtain

$$\sin\theta = \sin\alpha + \xi\cos\alpha; \qquad \cos\theta = \cos\alpha - \xi\sin\alpha.$$

Substituting this in eq. (c) and using eq. (d)

$$\ddot{\xi} - \frac{h^2}{\sin^3\alpha}\left\{\cos\alpha - \xi\left(\frac{3\cos^2\alpha}{\sin\alpha} + \sin\alpha\right)\right\} = -\frac{g}{l}(\sin\alpha + \xi\cos\alpha).$$

Assuming that the constant α is adjusted so that eq. (e) is satisfied, we obtain,

$$\ddot{\xi} + (1 + 3\cos^2\alpha)\omega^2\xi = 0,$$

from which it can be concluded that the oscillation ξ in the value of θ has the period

$$\frac{2\pi}{\omega\sqrt{1 + 3\cos^2\alpha}}.$$

When α is small this period approaches the value π/ω, i.e., nearly two oscillations occur for each revolution of the conical pendulum.

28. Free Vibrations. General Discussion.—If a system is disturbed from its position of stable equilibrium by an impact or by the application and sudden removal of an additional force the forces in the disturbed position will no longer be in equilibrium and vibrations will ensue. We consider first the case in which variable external forces are absent and *free vibrations* take place. Assuming that during vibration the system performs only small displacements, let $q_1, q_2 \cdots q_n$ be the generalized coordinates chosen in such a manner that they vanish when the system is in the position of equilibrium. Assuming now that the forces acting on the parts of the system are of the nature of elastic forces, their magnitudes will be homogeneous linear functions of the small displacements of the system, i.e., linear functions of the coordinates $q_1, q_2 \cdots q_n$. The potential energy of the system will then be a homogeneous function of the second degree of the same coordinates,

$$2V = c_{11}q_1^2 + c_{22}q_2^2 + \cdots + 2c_{12}q_1q_2 + \cdots. \tag{73}$$

The formula for the kinetic energy of the system is

$$2T = \sum m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)$$

or substituting here for the Cartesian coordinates their expressions in terms of the generalized coordinates (see eq. *a*, paragraph 25)

$$2T = a_{11}\dot{q}_1^2 + a_{22}\dot{q}_2^2 + \cdots + 2a_{12}\dot{q}_1\dot{q}_2 + \cdots. \tag{74}$$

In the general case the coefficients $a_{11}, a_{22}, \cdots$ will be functions of the coordinates $q_1, q_2, \cdots$ but in the case of small displacements they can be considered as constant and equal to their value at the position of equilibrium. Substituting now (73) and (74) in Lagrange's equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0, \tag{a}$$

the general equations of motion will be obtained.

Consider first the case of a system with two degrees of freedom only.

Then,

$$2V = c_{11}q_1^2 + c_{22}q_2^2 + 2c_{12}q_1q_2, \tag{b}$$

$$2T = a_{11}\dot{q}_1^2 + a_{22}\dot{q}_2^2 + 2a_{12}\dot{q}_1\dot{q}_2. \tag{c}$$

Substituting in eq. (a)

$$\begin{aligned} a_{11}\ddot{q}_1 + a_{12}\ddot{q}_2 + c_{11}q_1 + c_{12}q_2 &= 0, \\ a_{22}\ddot{q}_2 + a_{12}q_1 + c_{22}q_2 + c_{12}q_1 &= 0. \end{aligned} \tag{d}$$

The solutions of these two linear equations with constant coefficients can be taken in the following form,

$$q_1 = \lambda_1 \cos (pt + \alpha); \qquad q_2 = \lambda_2 \cos (pt + \alpha). \tag{e}$$

in which λ_1, λ_2, p and α denote constants which must be chosen so as to satisfy eqs. (d).

Substituting (e) in equations (d) we obtain

$$\begin{aligned} \lambda_1(a_{11}p^2 - c_{11}) + \lambda_2(a_{12}p^2 - c_{12}) &= 0, \\ \lambda_1(a_{12}p^2 - c_{12}) + \lambda_2(a_{22}p^2 - c_{22}) &= 0. \end{aligned} \tag{f}$$

Eliminating now λ_1 and λ_2 we get

$$(a_{11}p^2 - c_{11})(a_{22}p^2 - c_{22}) - (a_{12}p^2 - c_{12})^2 = 0. \tag{g}$$

This equation is a quadratic in p^2 and it can be shown that it has two real positive roots.

From the condition that the expressions (b) and (c) for the potential and the kinetic energy are essentially positive, the following relations between the coefficients entering in these expressions hold:

$$c_{11} > 0, \qquad c_{22} > 0, \qquad c_{12}^2 - c_{11}c_{22} < 0, \tag{h}$$

$$a_{11} > 0, \qquad a_{22} > 0, \qquad a_{12}^2 - a_{11}a_{22} < 0. \tag{k}$$

Substituting now in eq. (g) $p^2 = 0$ or $p^2 = +\infty$ and using (h) and (k) it can be concluded that the left side of this equation has a positive value. On the other hand, by taking $p^2 = (c_{11}/a_{11})$ or $p^2 = (c_{22}/a_{22})$ the left side of eq. (g) becomes negative. This means that between $p^2 = 0$ and $p^2 = +\infty$ the curve representing the left side of eq. (g) crosses the abscissa axis in two points, representing two positive roots for p^2. Let p_1^2 be one of these two roots. Substituting it in the first of eqs. (f) we have

$$\frac{\lambda_1}{a_{12}p_1^2 - c_{12}} = \frac{\lambda_2}{c_{11} - a_{11}p_1^2} = \mu_1, \text{ say.}$$

It is seen that for this particular root p_1^2 there is a definite ratio between the amplitudes λ_1 and λ_2 which determines the *mode* of vibration and the solution (e) becomes

$$q_1 = \mu_1(a_{12}p_1^2 - c_{12}) \cos (p_1t + \alpha_1); \qquad q_2 = \mu_1(c_{11} - a_{11}p_1^2) \cos (p_1t + \alpha_1).$$

Only the positive value for p_1 should be taken in this solution because the solution does not change when p_1 and α_1 change signs. The second root p_2^2 of equation (g) gives an analogous solution with the constants μ_2 and α_2. Combining these two solutions the general solution of the equations (d) will be obtained.

$$\begin{aligned} q_1 &= \mu_1(a_{12}p_1^2 - c_{12}) \cos (p_1t + \alpha_1) + \mu_2(a_{12}p_2^2 - c_{12}) \cos (p_2t + \alpha_2), \\ q_2 &= \mu_1(c_{11} - a_{11}p_1^2) \cos (p_1t + \alpha_1) + \mu_2(c_{11} - a_{11}p_2^2) \cos (p_2t + \alpha_2), \end{aligned} \qquad (l)$$

containing four arbitrary constants μ_1, μ_2, α_1 and α_2 which can be calculated when the initial values of the coordinates q_1 and q_2 and of the corresponding velocities $\dot{q}_1$ and $\dot{q}_2$ are given.

It is seen that in the case of a system with two degrees of freedom two *modes of vibration* are possible, corresponding to two different roots of the eq. (g) called the *frequency equation*. In each of these modes of vibration the generalized coordinates q_1 and q_2 are simple harmonic functions of the same period and the same phase. Each of these modes is called a *normal mode* of vibration. Its period is determined by the construction of the system and also its *type*, because the ratio between the amplitudes λ_1 and λ_2 is determined, only the absolute value of the amplitude being arbitrary. When a system oscillates in one of the normal modes of vibration every point performs a simple harmonic motion of the same period and the same phase; all parts of the system passing simultaneously through their respective equilibrium positions.

The generalized coordinates q_1 and q_2 determining the configuration of a system can be chosen in various ways; one particular choice is especially simple for analytical discussion. Assume that the coordinates are chosen in such a manner that the terms containing products of the coordinates and the corresponding velocities in the expressions (b) and (c) disappear, then,

$$\begin{aligned} 2V &= c_{11}q_1^2 + c_{22}q_2^2, \\ 2T &= a_{11}\dot{q}_1^2 + a_{22}\dot{q}_2^2. \end{aligned}$$

The corresponding equations of motion are

$$a_{11}\ddot{q}_1 + c_{11}q_1 = 0; \qquad a_{22}\ddot{q}_2 + c_{22}q_2 = 0;$$

we obtain two independent differential equations so that in each normal

mode of vibration only one coordinate is varying. Such coordinates are called *normal* or *principal coordinates* of the system.

In the general case of a system with n degrees of freedom substituting the expressions (73) and (74) for the potential and kinetic energies in Lagrange's equation (a) we obtain differential equations of motion such as

$$\begin{array}{l} a_{11}\ddot{q}_1 + a_{12}\ddot{q}_2 + a_{13}\ddot{q}_3 + \cdots + c_{11}q_1 + c_{12}q_2 + c_{13}q_3 + \cdots = 0, \\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_{n1}\ddot{q}_1 + a_{n2}\ddot{q}_2 + a_{n3}\ddot{q}_3 + \cdots + c_{n1}q_1 + c_{n2}q_2 + c_{n3}q_3 + \cdots = 0. \end{array} \qquad (m)$$

These simultaneous differential equations are linear and of the second order with constant coefficients. Particular solutions of these equations can be obtained by taking

$$q_1 = \lambda_1 \cos (pt + \alpha) \cdots, \qquad q_n = \lambda_n \cos (pt + \alpha).$$

Substituting in (m) we have

$$\begin{array}{l} \lambda_1(a_{11}p^2 - c_{11}) + \lambda_2(a_{12}p^2 - c_{12}) \cdots + \lambda_n(a_{1n}p^2 - c_{1n}) = 0, \\ \cdot \\ \lambda_1(a_{n1}p^2 - c_{n1}) + \lambda_2(a_{n2}p^2 - c_{n2}) + \cdots + \lambda_n(a_{nn}p^2 - c_{nn}) = 0. \end{array} \qquad (n)$$

Proceeding as in the case of a system with two degrees of freedom and eliminating $\lambda_1, \cdots \lambda_n$ from the equations (n) we arrive at the *frequency equation*

$$\Delta(p^2) = 0, \qquad (75)$$

where $\Delta(p^2)$ is the following determinant.

$$\begin{vmatrix} (a_{11}p^2 - c_{11}), & (a_{12}p^2 - c_{12}), & \cdots & (a_{1n}p^2 - c_{1n}) \\ \cdot & \cdot & \cdot & \cdot \\ (a_{n1}p^2 - c_{n1}), & (a_{n2}p^2 - c_{n2}), & \cdots & (a_{nn}p^2 - c_{nn}) \end{vmatrix}$$

Equation (75) is of the nth degree in p^2 and it can be shown * that all the n roots of this equation are real and positive provided we have vibration about a position of stable equilibrium of the system. Let p_s^2 be one of these roots. Substituting it in the equations (n) the $n - 1$ ratios

$$\lambda_1 : \lambda_2 : \lambda_3 : \cdots : \lambda_n$$

will be obtained and all amplitudes can be determined as functions of one arbitrary constant, say μ_s. The corresponding solution of the eq. (m) is

$$q_1 = \lambda_1 \cos (p_s t + \alpha_s), \cdots q_n = \lambda_n \cos (p_s t + \alpha_s). \qquad (o)$$

* See, for example, H. Lamb, Higher Mechanics, p. 222.

It contains two arbitrary constants μ_s and α_s and represents one of the *principal modes* of vibration. The *frequency* of this vibration, depending on the magnitude of p_s, and the *type* of vibration, depending on the ratios $\lambda_1 : \lambda_2 : \cdots$, are completely determined by the constitution of the system. During this vibration all particles of the system perform simple harmonic motions of the same period $2\pi/p_s$ and of the same phase α_s, passing simultaneously through their respective equilibrium positions.

The general solution of eq. (m) will be obtained by superimposing n principal modes of vibration, such as (o), corresponding to n different roots of the frequency equation (75).*

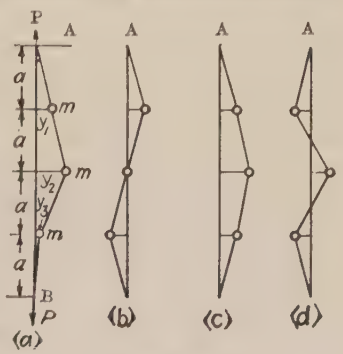


FIG. 68.

For illustrating this general theory a simple example of the vibration of a vertically stretched string AB with three equal and equidistant particles m will now be considered (Fig. 68, a). Assuming that the lateral deflections y of the string during the vibration are very small and neglecting the corresponding small fluctuations in the tensile force P, the potential energy of tension will be obtained by multiplying P with the elongation of the string.

$$V = Pa\left(\frac{1}{2}\frac{y_1^2}{a^2} + \frac{1}{2}\frac{(y_2 - y_1)^2}{a^2} + \frac{1}{2}\frac{(y_3 - y_2)^2}{a^2} + \frac{1}{2}\frac{y_3^2}{a^2}\right)$$
$$= \frac{P}{a}(y_1^2 + y_2^2 + y_3^2 - y_1y_2 - y_2y_3).$$

The kinetic energy of the system is

$$T = \frac{m}{2}(\dot{y}_1^2 + \dot{y}_2^2 + \dot{y}_3^2).$$

Substituting in Lagrange's eq. (71)

$$m\ddot{y}_1 + \frac{P}{a}(2y_1 - y_2) = 0,$$

$$m\ddot{y}_2 + \frac{P}{a}(2y_2 - y_1 - y_3) = 0, \qquad (r)$$

* For a particular case of equal roots of the frequency equation see, for example, H. Lamb, Higher Mechanics, p. 222.

$$m\ddot{y}_3 + \frac{P}{a}(2y_3 - y_2) = 0.$$

Assuming,

$$y_1 = \lambda_1 \cos(pt - \alpha); \qquad y_2 = \lambda_2 \cos(pt - \alpha); \qquad y_3 = \lambda_3 \cos(pt - \alpha).$$

and substituting in eqs. (r)

$$\begin{aligned} \lambda_1(p^2 - 2\beta) + \lambda_2\beta &= 0, \\ \lambda_1\beta + \lambda_2(p^2 - 2\beta) + \lambda_3\beta &= 0, \\ \lambda_2\beta + \lambda_3(p^2 - 2\beta) &= 0, \end{aligned} \tag{s}$$

where,

$$\beta = \frac{P}{ma}.$$

By calculating the determinant of the equations (s) we arrive at the following frequency equation

$$(p^2 - 2\beta)(p^4 - 4p^2\beta + 2\beta^2) = 0. \tag{t}$$

Substituting the root $p^2 = 2\beta$ of this equation in eqs. (s) we have,

$$\lambda_2 = 0 \qquad \text{and} \qquad \lambda_1 = -\lambda_3;$$

the corresponding type of vibration is represented in Fig. 68, b. The two other roots, $p^2 = (2 \pm \sqrt{2})\beta$, of the same eq. ($t$), substituted in eqs. ($s$) give us

$$\lambda_1 = \lambda_3 = \pm \frac{1}{\sqrt{2}}\lambda_2.$$

The corresponding types of vibration are shown in Fig. 68, c and d. The configuration (c), where all the particles are moving simultaneously in the same direction, represents the lowest or fundamental type of vibration, its period being the largest. The type (d) is the highest type of vibration to which corresponds the highest frequency.

29. Forced Vibrations.—In those cases where periodical disturbing forces are acting on the system forced vibrations will take place. By using Lagrange's equations in their general form (72) and substituting for T and V their general expressions (74) and (73) the equations of motion will be,

$$\begin{aligned} a_{11}\ddot{q}_1 + a_{12}\ddot{q}_2 + a_{13}\ddot{q}_3 + \cdots + c_{11}q_1 + c_{12}q_2 + c_{13}q_3 + \cdots &= Q_1, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}\ddot{q}_1 + a_{n2}\ddot{q}_2 + a_{n3}\ddot{q}_3 + \cdots + c_{n1}q_1 + c_{n2}q_2 + c_{n3}q_3 + \cdots &= Q_n. \end{aligned} \tag{a}$$

We proceed to consider now the most important case where the

generalized forces are of the simple harmonic type having the same period and the same phase so that every one of these forces can be represented in the form $Q_s = b_s \cos (mt + \beta)$, m and β being constant for any particular force.

A particular solution of eqs. (a) can be taken in the form:

$$q_1 = \lambda_1 \cos (mt + \beta); \qquad q_2 = \lambda_2 \cos (mt + \beta); \ldots q_n = \lambda_n \cos (mt + \beta).$$

Substituting in eqs. (a)

$$\begin{array}{l} (c_{11} - a_{11}m^2)\lambda_1 + (c_{12} - a_{12}m^2)\lambda_2 + \cdots + (c_{1n} - a_{1n}m^2)\lambda_n = b_1, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ (c_{n1} - a_{n1}m^2)\lambda_1 + (c_{n2} - a_{n2}m^2)\lambda_2 + \cdots + (c_{nn} - a_{nn}m^2)\lambda_n = b_n. \end{array} \tag{b}$$

From these equations the amplitudes $\lambda_1, \lambda_2, \cdots \lambda_n$ of the vibrations can be obtained.

It should be noted that when the determinant of the equations (b) approaches zero, i.e., when the period of the disturbing force approaches one of the natural periods of vibration, the amplitudes of vibration become very great. This is the principle of *resonance* which was discussed before for the case of systems with one degree of freedom.

If the generalized coordinates $q_1, q_2, \cdots q_n$ are *normal* or *principal coordinates* of the system, the expressions for the kinetic and potential energies become

$$2T = a_{11}\dot{q}_1^2 + a_{22}\dot{q}_2^2 + \cdots + a_{nn}\dot{q}_n^2,$$
$$2V = c_{11}q_1^2 + c_{22}q_2^2 + \cdots + c_{nn}q_n^2.$$

Substituting in Lagrange's equation (72)

$$\begin{array}{l} a_{11}\ddot{q}_1 + c_{11}q_1 = Q_1, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{nn}\ddot{q}_n + c_{nn}q_n = Q_n. \end{array} \tag{76}$$

Assuming again,

$$\begin{array}{l} Q_s = b_s \cos (mt + \beta), \\ q_1 = \lambda_1 \cos (mt + \beta), \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ q_n = \lambda_n \cos (mt + \beta), \end{array}$$

we have, from eqs. (76),

$$\lambda_s = \frac{b_s}{c_{ss} - m^2 a_{ss}} = \frac{b_s}{c_{ss}} \frac{1}{1 - \dfrac{m^2}{p^2}}.$$

Here b_s/c_{ss} represents the statical deflection produced by the force Q_s at the point of its application and m^2/p^2 the square of the ratio between the frequency of the force and the frequency of natural vibration. An analogous result has been previously obtained for systems with one degree of freedom (see eq. 20).

In our previous discussion friction was neglected. Assuming that there is a small friction and that the frictional forces are proportional to the generalized velocities it can be shown * that the ratios $\lambda_1 : \lambda_2 : \lambda_3 : \lambda_n$ and the frequencies p of the free vibrations will differ only slightly from the values which they would have if there were no friction and that the main effect of the small friction is on the amplitudes of vibration, this effect being of the same kind as previously shown for systems with one degree of freedom. The general theory developed will now be applied to the solution of several particular problems.

30. Vibrations of Vehicles.—*General Equations.*—The problem of the vibration of a four wheel vehicle as a system with many degrees of freedom is a very complicated one. In the following pages this problem is simplified and only the pitching motion in one plane † (Fig. 69) will be considered. In such a case the system has only *two degrees of freedom* and its position during the vibration can be specified by two coordinates: the vertical displacement z of the center of gravity C and the angle of rotation θ as shown in Fig. 69, *b*. Both of these coordinates will be measured from the position of equilibrium.

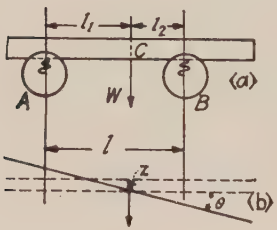


FIG. 69.

Let

W = sprung weight of the vehicle.

$I = (W/g)i^2$ = moment of inertia of the sprung mass about the axis through the center of gravity C.

i = radius of gyration.

k_1, k_2 = spring constants for the axles A and B, respectively.

l_1, l_2 = distances of the center of gravity from the same axes.‡

* See Lord Rayleigh's Theory of Sound, Vol. I.

† Rolling motion of the car is excluded from the following discussion.

‡ These distances are considered as constant in the further discussion.

Then the kinetic energy of motion will be

$$T = \frac{1}{2}\frac{W}{g}\dot{z}^2 + \frac{1}{2}\frac{W}{g}i^2\dot{\theta}^2. \tag{a}$$

In calculating the potential energy let δ_a, δ_b denote the initial deflections of the springs at the axles A and B, respectively, then,

$$\delta_a = \frac{Wl_2}{lk_1}, \qquad \delta_b = \frac{Wl_1}{lk_2}. \tag{b}$$

The increase in the potential energy of deformation of the springs during motion will be

$$V_1 = \frac{k_1}{2}\{(z - l_1\theta) + \delta_a\}^2 + \frac{k_2}{2}\{(z + l_2\theta) + \delta_b\}^2 - \frac{k_1\delta_a^2}{2} - \frac{k_2\delta_b^2}{2}$$

or by using (*b*)

$$V_1 = \frac{k_1}{2}(z - l_1\theta)^2 + \frac{k_2}{2}(z + l_2\theta)^2 + Wz.$$

The decrease in the potential energy of the system due to the lowering of the center of gravity will be,

$$V_2 = Wz.$$

The complete expression for the potential energy of the system during motion is therefore,

$$V = V_1 - V_2 = \frac{k_1}{2}(z - l_1\theta)^2 + \frac{k_2}{2}(z + l_2\theta)^2. \tag{c}$$

Substituting (*a*) and (*c*) in Lagrange's equations (71) the following equations for the *free vibrations* of the vehicle will be obtained

$$\frac{W}{g}\ddot{z} = -k_1(z - l_1\theta) - k_2(z + l_2\theta),$$

$$\frac{W}{g}i^2\ddot{\theta} = l_1k_1(z - l_1\theta) - l_2k_2(z + l_2\theta).$$

Letting

$$\frac{(k_1 + k_2)g}{W} = a; \qquad \frac{(-k_1l_1 + k_2l_2)g}{W} = b; \qquad \frac{(l_1^2k_1 + l_2^2k_2)g}{W} = c, \tag{d}$$

we have

$$\begin{aligned} \ddot{z} + az + b\theta &= 0, \\ \ddot{\theta} + \frac{b}{i^2}z + \frac{c}{i^2}\theta &= 0. \end{aligned} \tag{e}$$

These two simultaneous differential equations show that in general the coordinates z and θ are not independent of each other and if, for instance, in order to produce vibrations, the frame of the car be displaced parallel to itself in the z direction and then suddenly released, not only a vertical displacement z but also a rotation θ will take place during the subsequent vibration. The coordinates z and θ become independent only in the case when $b = 0$ in eqs. (e). This occurs when

$$k_1 l_1 = k_2 l_2, \tag{f}$$

i.e., when the spring constants are inversely proportional to the spring distances from the center of gravity. In such cases a load applied at the center of gravity will only produce vertical displacement of the frame without rotation. Such conditions exist in the case of railway carriages where usually $l_1 = l_2$ and $k_1 = k_2$.

Returning now to the general case we take the solution of the eqs. (e) in the following form

$$z = A \cos (pt + \alpha); \qquad \theta = B \cos (pt + \alpha).$$

Substituting in eqs. (e)

$$\begin{aligned} A(a - p^2) + bB &= 0, \\ \frac{b}{i^2} A + \left(\frac{c}{i^2} - p^2\right) B &= 0. \end{aligned} \tag{g}$$

Eliminating A and B from eqs. (g) the following frequency equation will be obtained,

$$(a - p^2)\left(\frac{c}{i^2} - p^2\right) - \frac{b^2}{i^2} = 0. \tag{h}$$

The two roots of (h) considered as an equation in p^2 are,

$$\begin{aligned} p^2 &= \frac{1}{2}\left(\frac{c}{i^2} + a\right) \pm \sqrt{\frac{1}{4}\left(\frac{c}{i^2} + a\right)^2 - \frac{ac}{i^2} + \frac{b^2}{i^2}} \\ &= \frac{1}{2}\left(\frac{c}{i^2} + a\right) \pm \sqrt{\frac{1}{4}\left(\frac{c}{i^2} - a\right)^2 + \frac{b^2}{i^2}}. \end{aligned} \tag{k}$$

Taking into consideration, from eq. (d), that

$$ac - b^2 = \frac{g^2}{W^2} k_1 k_2 (l_1 + l_2)^2,$$

it can be concluded that both roots of eq. (h) are real and positive.

Principal Modes of Vibration.—Substituting (k) in the first of the eqs. (g) the following values for the ratio A/B between the amplitudes will be obtained.

$$\frac{A}{B} = \frac{b}{p^2 - a} = \frac{b}{\frac{1}{2}\left(\frac{c}{i^2} - a\right) \pm \sqrt{\frac{1}{4}\left(\frac{c}{i^2} - a\right)^2 + \frac{b^2}{i^2}}}. \tag{l}$$

The + sign, as is seen from (k), corresponds to the mode of vibration having the higher frequency while the − sign corresponds to vibrations of lower frequency.

In the further discussion it will be assumed that

$$b > 0 \quad \text{or} \quad k_2 l_2 > k_1 l_1.$$

This means that under the action of its own weight the displacement of the car is such as shown in Fig. 70; the displacement in downwards direction is associated with a rotation in the direction of the negative θ. Under this assumption the amplitudes A and B will have opposite signs

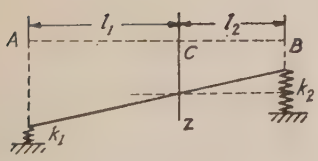


Fig. 70.

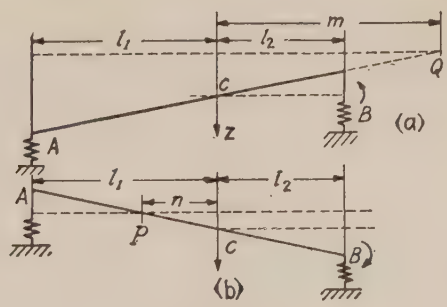


Fig. 71.

if the negative sign be taken before the radical in the denominator of (l) and they will have the same signs when the positive sign be taken. The corresponding two types of vibration are shown in Fig. 71. The type (a) has a lower frequency and can be considered as a rotation about a certain point Q to the right of the center of gravity C. The type (b) having a higher frequency, consists of a rotation about a certain point P to the left of C. The distances m and n of the points Q and P from the center of gravity are given by the absolute values of the right side of eq. (l) and we obtain a very simple relation,

$$mn = -\frac{b}{\frac{1}{2}\left(\frac{c}{i^2} - a\right) + \sqrt{\frac{1}{4}\left(\frac{c}{i^2} - a\right)^2 + \frac{b^2}{i^2}}} \cdot \frac{b}{\frac{1}{2}\left(\frac{c}{i^2} - a\right) - \sqrt{\frac{1}{4}\left(\frac{c}{i^2} - a\right)^2 + \frac{b^2}{i^2}}} = i^2. \tag{m}$$

In the particular case, when $b = 0$, i.e., $k_1l_1 = k_2l_2$ the distance n becomes equal to zero and m becomes infinitely large. This means that in this case one of the principal modes of vibration consists of a rotation about the center of gravity and the other consists of a translatory movement without rotation. A vertical load applied at the center of gravity in this case will produce only a vertical displacement and both springs will get equal compressions.

If, in addition to b, $(c/i^2) - a$ becomes equal to zero, both frequencies, as given by eq. (k), become equal and the two types of vibration will have the same period.

Numerical Example.—A numerical example of the above theory will now be considered.* Taking a case with the following data: $W = 966$ lbs.; $i^2 = 13$ ft.^{2}; $l_1 = 4$ ft.; $l_2 = 5$ ft.; $k_1 = 1600$ lbs./ft.; $k_2 = 2400$ lb./ft.

These stiffnesses correspond to static deflections (see eq. b)

$$\delta_a = 4.0 \text{ in.}, \qquad \delta_b = 2.15 \text{ in.}$$

From eqs. (d)

$$a = 133.3, \qquad b = 186.7, \qquad c = 2853.$$

Substituting in (k) we obtain the following two roots $p_1^2 = 109$, $p_2^2 = 244$. The corresponding frequencies are

$p_1 = 10.5$ radians per second and $p_2 = 15.6$ radians per second, respectively, or

$N_1 = 100$ and $N_2 = 150$ complete oscillations per minute.

From eq. (l) we have

$$\frac{A}{B} = -7.71 \text{ ft.} \qquad \text{or} \qquad \frac{A}{B} = 1.69 \text{ ft.}$$

This means that in the slower mode of vibration the sprung weight oscillates 7.71 ft. per radian of pitching motion or 1.62 inches per degree.

* See the paper by H. S. Rowell, Proc. Inst. Automobile Engineers, London, Vol. XVII, Part II, p. 455 (1923).

In the higher mode of vibration the sprung weight oscillates 1.69 ft. for every radian of pitching motion or .355 inch per degree.

Roughly speaking in the slower mode of vibration the car is bouncing, the deflections of two springs being of the same sign and in the ratio

$$\frac{\delta_b'}{\delta_a'} = \frac{7.71 - 5}{7.71 + 4} = .23.$$

In the quicker mode of vibration the car is mostly pitching.

It is interesting to note that a good approximation for the frequencies of the principal modes of vibration can be obtained by using the theory of a system with one degree of freedom. Assuming first that the spring at B (See Fig. 69) is removed so that the car can bounce on the spring A about the axis B as a hinge. Then the equation of motion is

$$\left(I + \frac{W}{g} l_2^2\right) \ddot{\theta} + k_1 l^2 \theta = 0,$$

so that the "constrained" frequency is

$$p_1' = l \sqrt{\frac{k_1}{I + \frac{W}{g} l_2^2}},$$

or substituting the numerical data of the above example

$$p_1' = 9 \sqrt{\frac{1600}{\frac{966}{32.2}(13 + 25)}} = 10.7.$$

This is in good agreement with the frequency 10.5 obtained above for the lower type of vibration of the car. In the same manner considering the bouncing of the car on the spring B about the axis A as a hinge, we obtain $p_2' = 15.0$ as compared with $p_2 = 15.6$ given above for the quicker mode of vibration.

On the basis of this a practical method for obtaining the frequencies of the principal modes of vibration by test is to lock the front springs and bounce the car; then lock the rear springs and again bounce the car. The frequencies obtained by these tests will represent a good approximation.

Beating Phenomena.—Returning now to the general solution of the eqs. (e) and denoting by p_1 and p_2, the two roots obtained from (k) we have

$$\begin{aligned} z &= A_1 \cos (p_1 t + \alpha_1) + A_2 \cos (p_2 t + \alpha_2), \\ \theta &= B_1 \cos (p_1 t + \alpha_1) + B_2 \cos (p_2 t + \alpha_2), \end{aligned} \qquad (r)$$

in which (see eq. l)

$$\frac{A_1}{B_1} = \frac{b}{p_1^2 - a}; \qquad \frac{A_2}{B_2} = \frac{b}{p_2^2 - a}. \tag{s}$$

The general solution (r) contains four arbitrary constants A_1, A_2, α_1 and α_2, which must be determined for every particular case so as to satisfy the initial conditions. Assume, for instance, that in the initial moment a displacement λ exists in a downward direction without rotation and that the car is then suddenly released. In such a case the initial conditions are

$$(z)_{t=0} = \lambda; \qquad (\dot{z})_{t=0} = 0; \qquad (\theta)_{t=0} = 0; \qquad (\dot{\theta})_{t=0} = 0.$$

These conditions will be satisfied by taking in eqs. (r)

$$\begin{aligned} &\alpha_1 = \alpha_2 = 0, \\ &A_1 = \lambda \frac{a - p_2^2}{p_1^2 - p_2^2}; \qquad A_2 = \lambda \frac{p_1^2 - a}{p_1^2 - p_2^2}; \\ &B_1 = A_1 \frac{(p_1^2 - a)}{b}; \qquad B_2 = A_2 \frac{(p_2^2 - a)}{b}. \end{aligned} \tag{t}$$

We see that under the assumed conditions both modes of vibration will be produced which at the beginning will be in the same phase but with elapse of time, due to the difference in frequencies, they will become displaced with respect to each other and a complicated combined motion will take place. If the difference of frequencies is a very small one the characteristic "*beating phenomenon*," i.e., vibrations with periodically varying amplitude, will take place. In considering this particular case, assume in eq. (k) that

$$\frac{c}{i^2} - a = 0 \qquad \text{and} \qquad \frac{b}{i} = \delta,$$

where δ is a small quantity. Then

$$p_1^2 = a - \delta; \qquad p_2^2 = a + \delta,$$

and from (t) we obtain,

$$A_1 = \frac{\lambda}{2}; \qquad A_2 = \frac{\lambda}{2}; \qquad B_1 = -\frac{\lambda}{2i}; \qquad B_2 = \frac{\lambda}{2i}.$$

Solution (n) becomes

$$\begin{aligned} z &= \frac{\lambda}{2}(\cos p_1 t + \cos p_2 t) = \lambda \cos \frac{p_1 + p_2}{2} t \cos \frac{p_1 - p_2}{2} t, \\ \theta &= \frac{\lambda}{2i}(-\cos p_1 t + \cos p_2 t) = \frac{\lambda}{i} \sin \frac{p_1 + p_2}{2} t \sin \frac{p_1 - p_2}{2} t. \end{aligned} \tag{u}$$

Due to the fact that $p_1 - p_2$ is a small quantity the functions $\cos\{(p_1 + p_2)/2\}t$ and $\sin\{(p_1 + p_2)/2\}t$ will be quickly varying functions so that they will perform several cycles before the slowly varying function $\sin\{(p_1 - p_2)/2\}t$ or $\cos\{(p_1 - p_2)/2\}t$ can undergo considerable change. As a result, oscillations with periodically varying amplitudes will be obtained (see Fig. 9).

Forced Vibrations.—The disturbing forces producing forced oscillations of a car are transmitted by the springs. In the general discussion above it was shown that the two principal modes of vibration are oscillations about two definite points P and Q (Fig. 71). The corresponding generalized forces in such a case are the moments of the spring forces about the points P and Q. From this it can be concluded that any fluctuation in a spring force, produced by some kind of unevenness of the road, will produce simultaneously both types of vibrations provided that this spring force does not pass through one of the points P or Q. Assume, for instance, that the front wheels of a moving car encounter an obstacle on the road, the corresponding compression of the front springs will produce vibrations of the car. Now when the rear wheels reach the same obstacle, an additional impulse will be given to the oscillating car. The oscillations produced by this new impulse will be superimposed on the previous oscillations and the resulting motion will depend on the value t of the interval of time between the two impulses, or, denoting by v the velocity of the car, on the magnitude of l/v. It is easy to see that at a certain value of v the effects of the two impulses will be added and we will get very unfavorable conditions for these *critical speeds.* Let τ_1 and τ_2 denote the periods of the two principal modes of vibration and assume the interval $t = (l/v)$ be a multiple of these periods, so that

$$t = m_1\tau_1 = m_2\tau_2,$$

where m_1 and m_2 are integer numbers. Then the impulses will repeat after an integer number of oscillations and *resonance conditions* will take place.* Under such conditions large oscillations may be produced if there is not enough friction in the springs.

From this discussion it is clear that an arrangement where an impulse produced by one spring does not affect the other spring may be of practical interest. This condition will be satisfied when the body of the car can be replaced by a dynamical model with two masses W_1 and W_2 (Fig. 72)

* See P. Lemaire, La Technique Moderne, January 1921. See also the paper by H. S. Rowell, p. 481, mentioned above.

concentrated at the springs A and B. In this case we have

$$w_1 = \frac{Wl_2}{l}; \qquad w_2 = \frac{Wl_1}{l},$$

$$W_1l_1^2 + W_2l_2^2 = Wi^2,$$

from which

$$l_1l_2 = i^2. \tag{77}$$

Comparing with eq. (m) it can be concluded that the points P and Q (see Fig. 71) coincide in this case with the points A and B so that the fluctuations in the spring forces will be independent of each other and the condition of resonance will be excluded. It should be noted that when $l_1 = l_2$ condition (77) coincides with the rule given by Prof. H. Reissner that the radius of gyration of the sprung mass should be half the wheel base. In most of the modern cars the wheel base is larger than that given by eq. (77). This discrepancy should be attributed to steering and skidding conditions which necessitate an increase in wheel base.

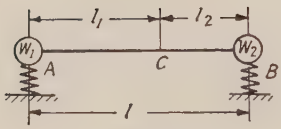


FIG. 72.

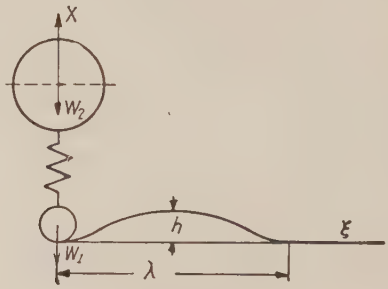


FIG. 73.

Pressure on the Road.—Due to dynamical causes the pressure of a wheel on the road during motion will be usually different from what we would have in the statical condition. Assuming the simple case illustrated in Fig. 72, the pressure of the wheel can be found from a consideration of the motion of the system, shown in Fig. 73, in which W_1 = weight directly transmitted on the road,* W_2 = spring borne weight, v = constant velocity of the motion of the wheel along the horizontal axis, x_1, x_2 = displacements in an upwards direction of the weights W_1 and W_2 from their position of equilibrium shown in Fig. 73. If there is no unevenness of the road, no vibration will take place during motion and the pressure on the road will be equal to the statical. Assume now that

* Spring effect of the tire is neglected in this discussion.

the road contour is rigid and can be represented by the equation:

$$y = \frac{h}{2}\left(1 - \cos\frac{2\pi\xi}{\lambda}\right),$$

where ξ is measured along the horizontal axis and λ is the wave length. Then, when moving with a constant velocity v along these waves the vertical displacements of the wheel considered as rigid will be represented by the equation

$$x_1 = \frac{h}{2}\left(1 - \cos\frac{2\pi vt}{\lambda}\right). \qquad (a)$$

The corresponding acceleration in a vertical direction is

$$\ddot{x} = \frac{h}{2}\frac{4\pi^2 v^2}{\lambda^2}\cos\frac{2\pi vt}{\lambda}.$$

Adding the inertia force to the weight the pressure on the road due to the unsprung mass alone will be

$$W_1 + \frac{W_1}{g}\frac{h}{2}\frac{4\pi^2 v^2}{\lambda^2}\cos\frac{2\pi vt}{\lambda}. \qquad (b)$$

The maximum pressure occurs when the wheel occupies the lowest position on the contour and is equal to

$$W_1 + \frac{W_1}{g}\frac{h}{2}\frac{4\pi^2 v^2}{\lambda^2}.$$

It is seen that the *dynamical effect* due to the inertia force increases as the square of the speed.

In order to obtain the complete pressure on the road, the pressure due to the spring force must be added to pressure (b) calculated above. This force will be given by the expression

$$W_2 - k(x_2 - x_1), \qquad (c)$$

in which the second term represents the change in the force of the spring due to the relative displacement $x_2 - x_1$ of the masses W_1 and W_2. This displacement can be obtained from the differential equation

$$\frac{W_2}{g}\ddot{x}_2 + k(x_2 - x_1) = 0, \qquad (d)$$

representing the equation of motion of the sprung weight W_2.

Substituting (a) for x_1 we have

$$\frac{W_2}{g}\ddot{x}_2 + kx_2 = \frac{kh}{2}\left(1 - \cos\frac{2\pi vt}{\lambda}\right). \qquad (e)$$

This equation represents vibration of the sprung weight produced by the wavy contour of the road. Assuming that at the beginning of the motion $x_1 = x_2 = 0$ and $\dot{x}_1 = \dot{x}_2 = 0$, the solution of eq. ($e$) will be

$$x_2 = \frac{h}{2}\left(1 + \frac{\tau_1^2}{\tau_2^2 - \tau_1^2}\cos\frac{2\pi t}{\tau_1} - \frac{\tau_2^2}{\tau_2^2 - \tau_1^2}\cos\frac{2\pi t}{\tau_2}\right), \qquad (f)$$

in which

$\tau_1 = 2\pi\sqrt{(W_2/kg)}$ natural period of vibration of the sprung weight,

$\tau_2 = (\lambda/v)$ time necessary to cross the wave length λ.

The force in the spring, from eqs. (a) and (c), is

$$W_2 - \frac{kh}{2}\,\frac{\tau_1^2}{\tau_2^2 - \tau_1^2}\left(\cos\frac{2\pi t}{\tau_1} - \cos\frac{2\pi t}{\tau_2}\right). \qquad (g)$$

Now, from (b) and (g), the pressure on the road in addition to the statical pressure will be

$$\frac{W_1}{g}\,\frac{h}{2}\,\frac{4\pi^2}{\tau_2}\cos\frac{2\pi t}{\tau_2^2} - \frac{kh}{2}\,\frac{\tau_1^2}{\tau_2^2 - \tau_1^2}\left(\cos\frac{2\pi t}{\tau_1} - \cos\frac{2\pi t}{\tau_2}\right). \qquad (h)$$

The importance of the first term increases with the speed while the second term becomes important under conditions of resonance. On this basis it can be concluded that with a good road surface and high speed the unsprung mass decides the road pressure and in the case of a rough road the sprung mass becomes important.

31. Torsional Vibrations of Shafts and Geared Systems.—*General.*—In the previous discussion of torsional vibrations (see paragraph 2) the simple case of a shaft with two rotating masses at the ends was considered. In the following the more general case of the vibration of a shaft with several rotating masses will be discussed (see Fig. 74). To such a system many problems on torsional vibrations in electric machinery, Diesel engines and propeller shafts can be reduced.* Let, I_1, I_2, I_3, $\cdots$ = moments of inertia of the rotating masses about the axis of the shaft, φ_1, φ_2, φ_3, $\cdots$ = angles of rotation of these masses during vibration, k_1, k_2, k_3, $\cdots$ = spring constants of the shaft, for the lengths ab, bc and cd, respectively. Then $k_1(\varphi_1 - \varphi_2)$, $k_2(\varphi_2 - \varphi_3)$ $\cdots$ represent the

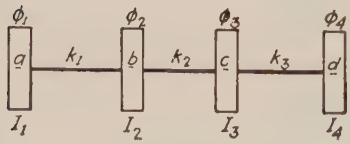


Fig. 74.

* The bibliography of this subject can be found in the very complete investigation of torsional vibration in the Diesel engine made by F. M. Lewis; see Trans. Soc. of Naval Architects and Marine Engineers, Vol. 33, 1925, p. 109, New York.

torque moments for the same lengths. Neglecting the moment of inertia of the rotating shaft in comparison with the moments of inertia $I_1, I_2, \cdots$ of the rotating masses, the kinetic energy of the vibrating system will be,

$$T = \tfrac{1}{2}I_1\dot{\varphi}_1^2 + \tfrac{1}{2}I_2\dot{\varphi}_2^2 + \tfrac{1}{2}I_3\dot{\varphi}_3^2 + \cdots. \qquad (a)$$

The potential energy of the system consisting in this case of the energy of twist is,

$$V = \tfrac{1}{2}k_1(\varphi_1 - \varphi_2)^2 + \tfrac{1}{2}k_2(\varphi_2 - \varphi_3)^2 + \tfrac{1}{2}k_3(\varphi_3 - \varphi_4)^2 + \cdots. \qquad (b)$$

Substituting (a) and (b) in Lagrange's eq. (71) the following differential equations for the free vibration for the the shaft will be obtained

$$\begin{aligned}
&I_1\ddot{\varphi}_1 + k_1(\varphi_1 - \varphi_2) = 0,\\
&I_2\ddot{\varphi}_2 + k_2(\varphi_2 - \varphi_3) - k_1(\varphi_1 - \varphi_2) = 0,\\
&I_3\ddot{\varphi}_3 + k_3(\varphi_3 - \varphi_4) - k_2(\varphi_2 - \varphi_3) = 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&I_{n-1}\ddot{\varphi}_{n-1} + k_{n-1}(\varphi_{n-1} - \varphi_n) - k_{n-2}(\varphi_{n-2} - \varphi_{n-1}) = 0,\\
&I_n\ddot{\varphi}_n - k_{n-1}(\varphi_{n-1} - \varphi_n) = 0.
\end{aligned} \qquad (c)$$

Adding these equations together we get

$$I_1\ddot{\varphi}_1 + I_2\ddot{\varphi}_2 + \cdots + I_n\ddot{\varphi}_n = 0,$$

which means that the moment of momentum of the system about the axis of the shaft remains constant during the free vibration. In the following this moment of momentum will be taken equal to zero. In this manner any rotation of the shaft as a rigid body will be excluded and only vibratory motion due to twist of the shaft will be considered.

Applying now the general method (see p. 124) and substituting in eqs. (c)

$$\varphi_1 = \lambda_1 \cos(pt + \alpha), \qquad \varphi_2 = \lambda_2 \cos(pt + \alpha), \cdots,$$

we obtain,

$$\begin{aligned}
&I_1\lambda_1 p^2 - k_1(\lambda_1 - \lambda_2) = 0,\\
&I_2\lambda_2 p^2 + k_1(\lambda_1 - \lambda_2) - k_2(\lambda_2 - \lambda_3) = 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&I_n\lambda_n p^2 + k_{n-1}(\lambda_{n-1} - \lambda_n) = 0.
\end{aligned} \qquad (d)$$

Eliminating from these equations $\lambda_1, \lambda_2, \cdots$ the *frequency equation* for calculating p^2 will be obtained.

The Case of Three Discs.—Let us consider first the simple case of three rotating masses. The system of eqs. (*d*) in this case becomes,

$$\begin{aligned} I_1\lambda_1 p^2 - k_1(\lambda_1 - \lambda_2) &= 0, \\ I_2\lambda_2 p^2 + k_1(\lambda_1 - \lambda_2) - k_2(\lambda_2 - \lambda_3) &= 0, \\ I_3\lambda_3 p^2 + k_2(\lambda_2 - \lambda_3) &= 0. \end{aligned} \tag{e}$$

Adding these equations we have

$$I_1\lambda_1 + I_2\lambda_2 + I_3\lambda_3 = 0. \tag{f}$$

From the first and the third of eqs. (*e*)

$$\lambda_1 = -\frac{k_1\lambda_2}{I_1p^2 - k_1}, \qquad \lambda_3 = -\frac{k_2\lambda_2}{I_3p^2 - k_2}. \tag{g}$$

Substituting in eq. (*f*) the following frequency equation will be obtained,

$$\frac{I_1I_2I_3}{k_1k_2}p^4 - \left(\frac{I_1I_2 + I_1I_3}{k_1} + \frac{I_2I_3 + I_1I_3}{k_2}\right)p^2 + (I_1 + I_2 + I_3) = 0. \tag{77}$$

This is a quadratic equation in p^2 and two roots p_1^2 and p_2^2, corresponding to the two principal modes of vibration can easily be calculated. Substituting p_1^2 and p_2^2 in eqs. (*g*) the values of the ratios λ_1/λ_2 and λ_2/λ_3 for the two principal modes of vibration will be obtained * and in this way the corresponding configurations of the system during the vibration can be established. The two modes of vibration are diagrammatically represented in Fig. 75. The lowest mode of vibration having one node is given in Fig. 75, *a* and the higher mode of vibration with two nodes in Fig. 75, *b*.

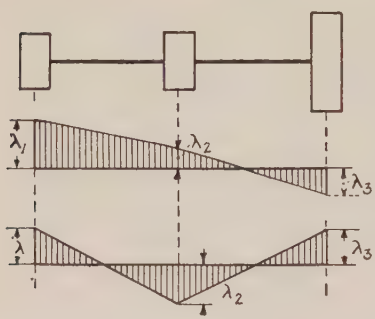


FIG. 75.

The Case of Many Discs.—In the general case the frequency equation will be obtained by equating to zero the determinant of eqs. (*d*) (see eq. 75, p. 124). Taking, for instance, the case of four rotating masses we

* Frequency equation for the case of four discs, see Hütte, Vol. I, p. 409, 25th ed. (1925).

obtain

$$\begin{vmatrix} I_1p^2 - k_1 & k_1 & 0 & 0 \\ k_1 & I_2p^2 - k_1 - k_2 & k_2 & 0 \\ 0 & k_2 & I_3p^2 - k_2 - k_3 & k_3 \\ 0 & 0 & k_3 & I_4p^2 - k_3 \end{vmatrix} = 0.$$

Dividing the rows of this determinant with I_1, I_2, I_3 and I_4, respectively, and using the notations,

$$p^2 = x, \qquad \frac{k_1}{I_1} = x_1 \qquad \frac{k_1 + k_2}{I_2} = x_2, \qquad \frac{k_2 + k_3}{I_3} = x_3, \qquad \frac{k_3}{I_4} = x_4,$$

$$\frac{k_1}{I_2} = n_{12}; \qquad \frac{k_2}{I_2} = n_{22}; \qquad \frac{k_2}{I_3} = n_{23}; \qquad \frac{k_3}{I_3} = n_{33}.$$

the frequency equation becomes

$$\begin{vmatrix} x - x_1 & x_1 & 0 & 0 \\ n_{12} & x - x_2 & n_{22} & 0 \\ 0 & n_{23} & x - x_3 & n_{33} \\ 0 & 0 & x_4 & x - x_4 \end{vmatrix} = 0.$$

Developing this determinant in the usual manner we obtain,

$$(x - x_1)\begin{vmatrix} x - x_2 & n_{22} & 0 \\ n_{23} & x - x_3 & n_{33} \\ 0 & x_4 & x - x_4 \end{vmatrix} - x_1\begin{vmatrix} n_{12} & n_{22} & 0 \\ 0 & x - x_3 & n_{33} \\ 0 & x_4 & x - x_4 \end{vmatrix} = 0$$

or

$$(x - x_1)(x - x_2)\begin{vmatrix} x - x_3 & n_{33} \\ x_4 & x - x_4 \end{vmatrix} -$$

$$- (x - x_1)n_{22}\cdot\begin{vmatrix} n_{23} & n_{33} \\ 0 & x - x_4 \end{vmatrix} - x_1n_{12}\begin{vmatrix} x - x_3 & n_{33} \\ x_4 & x - x_4 \end{vmatrix} = 0,$$

from which

$$(x - x_1)(x - x_2)(x - x_3)(x - x_4) - (x - x_1)(x - x_2)x_4n_{33} -$$
$$- (x - x_1)(x - x_4)n_{22}n_{23} - (x - x_3)(x - x_4)x_1n_{12} + x_1n_{12}n_{33}x_4 = 0. \quad (h)$$

Taking into consideration that

$$x_2 = n_{12} + n_{22}, \qquad x_3 = n_{23} + n_{33}.$$

the constant term in eq. (h) becomes equal to zero and one of the roots also becomes equal to zero. This result is due to the freedom of the

shaft to rotate as a rigid body about its axis. The three remaining roots of eq. (h), different from zero, give us the frequencies of the three principal modes of vibration of the system under consideration.

Numerical Example.—As an example the following numerical data will now be taken.*

$I_1 = 302$ lb. in. sec^2, $\quad k_1 = 316 \times 10^6$ lb. in. per radian,
$I_2 = 87500$ lb. in. sec^2, $\quad k_2 = 114.5 \times 10^6$ lb. in. per radian,
$I_3 = 1200$ lb. in. sec^2, $\quad k_3 = 1.09 \times 10^6$ lb. in. per radian.
$I_4 = .373$ lb. in. sec^2,

Substituting these data in eq. (h) above and using from now on the new notation $x = .0001\ p^2$ we obtain

$$x^3 - 407\ x^2 + 34492\ x - 296230 = 0. \tag{k}$$

In calculating the roots of equations of higher degree, such as the above, Horner's scheme is very useful.† This scheme represents the simplest way of calculating the numerical value of an integer rational function such as

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \tag{l}$$

for any given value k of the argument x. Instead of calculating the values k^n, k^{n-1}, $\cdots$ multiplying them by a_n, a_{n-1}, $\cdots$ and then adding them together the following procedure will be adopted. We multiply a_n with k and add $a_n k$ to a_{n-1}. Let a_{n-1}' denote the sum $a_n k + a_{n-1}$. Now multiplying a_{n-1}' with k and adding to a_{n-2} we obtain the sum $a_{n-2}' = a_{n-1}' k + a_{n-2}$. Proceeding further in this manner $a_{n-3}' = a_{n-2}' k + a_{n-3}$ will be obtained and finally $a_1' = a_2' k + a_1$ and $a_0' = a_1' k + a_0$. All these calculations can be put in the form of the following table.

$$\begin{array}{cccccc} a_n & a_{n-1} & a_{n-2} \cdots & a_2 & a_1 & a_0 \\ & a_n k & a_{n-1}' k \cdots & a_3' k & a_2' k & a_1' k \\ \hline a_n & a_{n-1}' & a_{n-2}' \cdots & a_2' & a_1' & \boxed{a_0'} \end{array} \tag{m}$$

It is easy to show that the value of a_0', calculated in this manner, represents the value $f(k)$ of the function (l) when $x = k$. The same scheme of calculation can be used also for developing $f(x)$ in a Taylor's series of

* These numerical data represent an actual case of a synchronous condenser calculated by J. Ormondroyd, Research Engineer, Westinghouse Elec. & Mfg. Co., East Pittsburgh, Pa.

† For approximate methods for calculating the roots of algebraic equations of higher degree, see "Practical Analysis"; by v. Sanden.

the form

$$f(x) = a_0' + A_1(x - k) + A_2(x - k)^2 + \cdots + A_n(x - k)^n,$$

in which

$$a_0' = f(k); \qquad A_1 = f'(k); \qquad A_2 = \frac{f''(k)}{1.2}; \cdots A_n = \frac{f^{(n)}(k)}{1.2.3 \cdots n}.$$

Take, for instance,

$$f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0.$$

In order to represent this function in the form of the series,

$$f(x) = a_0' + A_1(x - k) + A_2(x - k)^2 + A_3(x - k)^3 + A_4(x - k)^4 \quad (n)$$

we begin the calculations exactly in the same manner as explained above in the table (m) and arrive at the value $f(k) = a_0'$ given in the table (o) below

	a_4	a_3	a_2	a_1	a_0	
I		ka_4	ka_3'	ka_2'	ka_1'	I
	a_4	a_3'	a_2'	a_1'	a_0'	
II		ka_4	ka_3''	ka_2''		II
	a_4	a_3''	a_2''	$a_1'' = A_1$		(o)
III		ka_4	ka_3'''			III
	a_4	a_3'''	$a_2''' = A_2$			
IV		ka_4				IV
	$a_4 = A_4$	$a_3'''' = A_3$				

With the numbers $a_4, a_3', a_2', \cdots a_0'$ of the third line in this table, we proceed exactly in the same manner as with the numbers $a_4, a_3, a_2, \cdots$ of the first line, only the last number a_0' should not be taken into account. In this manner the numbers under the line II–II are obtained. The last of these numbers, namely a_1'', represents the coefficient A_1 in the series (n). Proceeding in the same manner we arrive at the values $a_2''' = A_2$, $a_3'''' = A_3$ and $a_4 = A_4$. In this manner all the coefficients of the series (n) have been calculated. The scheme (o) can now be applied for calculating the real roots of an algebraic equation with real coefficients, such as

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0. \qquad (p)$$

Let k be a rough approximation of the root to be calculated. Then proceeding as explained in table (o) and letting $x - k = u$, the following

equation will be obtained:

$$f(x) = A_n u^n + A_{n-1} u^{n-1} + \cdots + A_1 u + a_0' = 0, \qquad (q)$$

in which u is the difference between the true value of the root and its first approximation k. Assuming now that this difference is small and neglecting the terms with higher powers of u in eq. (q) we obtain as an approximation,

$$u_1 = -\frac{a_0'}{A_1}$$

and the second approximation for the root of eq. (p) will be

$$x = k + u_1.$$

In order to get a further approximation we proceed with eq. (q) exactly in the same manner as was done with eq. (p). Letting

$$u - u_1 = v$$

and following again the scheme of calculations indicated in table (o) the following equation for v will be obtained:

$$B_n v^n + B_{n-1} v^{n-1} + \cdots + B_1 v + a_0'' = 0$$

As a further correction we have

$$v_1 = -\frac{a_0''}{B_1}$$

and the third approximation for x will be

$$x = k + u_1 + v_1.$$

Proceeding in this manner a root of eq. (p) can be calculated with any desired accuracy.

Applying this method to the solution of the numerical equation (k) and taking as a first approximation $x = 9$, the following table will be obtained,

1	− 407	34492	− 296230
	9	− 3582	278190
1	− 398	30910	− 18040
	9	− 3501	
1	− 389	27409	
	9		
1	− 380		

The equation for the first correction $u = x - 9$ will be

$$u^3 - 380\,u^2 + 27409\,u - 18040 = 0. \qquad (q)$$

Taking as an approximation of this correction,

$$u_1 = \frac{18040}{27409} = \backsim .6,$$

and proceeding with eq. (q) as before, the following table will be obtained,

1	− 380	27409	− 18040
	.6	− 227.6	16309
1	− 379.4	27181	− 1731
	.6	227	
1	− 378.8	26954	

The equation for the second correction will be

$$v^3 - 378.8v^2 + 26954v - 1731 = 0,$$

from which

$$v_1 = \frac{1731}{26954} = \backsim .064.$$

Hence the root of equation (k) can be taken equal to

$$x_1 = 9 + u_1 + v_1 = 9.664.$$

The two other roots of the same equation will be

$$x_2 = 105; \qquad x_3 = 292.$$

The corresponding frequencies will be,

$$\frac{100\sqrt{x_1}}{2\pi} = 49.5 \text{ per sec.}; \quad \frac{100\sqrt{x_2}}{2\pi} = 163 \text{ per sec.}; \quad \frac{100\sqrt{x_3}}{2\pi} = 272 \text{ per sec.}$$

It is interesting to note that in the numerical example taken above the moments of inertia I_1 and I_3 are small in comparison with I_2; furthermore I_4 is small in comparison with I_3. In such a case a good approximation for the frequencies will be obtained by using the simple eq. 12 (see p. 6) derived for the case of a single disc. Considering I_2 as infinitely large and neglecting I_4 we will find for the torsional vibration of the disc I_3 the frequency

$$\frac{1}{2\pi}\sqrt{\frac{k_2}{I_3}} = \frac{1000}{2\pi}\sqrt{\frac{114.5}{1200}} = 49.3 \text{ per sec.},$$

which is very close to the frequency 49.5 found above for the fundamental type of vibration of the system with four discs.

Considering in the same manner the vibration of the disc I_1 when I_2 is considered as infinitely large and the vibration of I_4 when I_3 is considered as infinitely large we find the following values for the frequencies:

$$\frac{1}{2\pi}\sqrt{\frac{k_1}{I_1}} = \frac{1000}{2\pi}\sqrt{\frac{316}{302}} = 163 \text{ per sec.}$$

and

$$\frac{1}{2\pi}\sqrt{\frac{k_3}{I_3}} = \frac{1000}{2\pi}\sqrt{\frac{1.09}{.373}} = 272 \text{ per sec.,}$$

which coincide with the values calculated above. Frequently an analogous simplification of vibration problems is possible in the approximate calculation of frequencies. Such a simplification is especially important in cases of complicated systems with a large number of rotating masses as occur for instance in Diesel engines. Consider, as an example, the system shown in Fig. 76, where the moments of inertia of the generator, flywheel, of six cylinders and two air pumps, and also the distances between these masses are given.* The shaft is replaced by an *equivalent shaft* of uniform

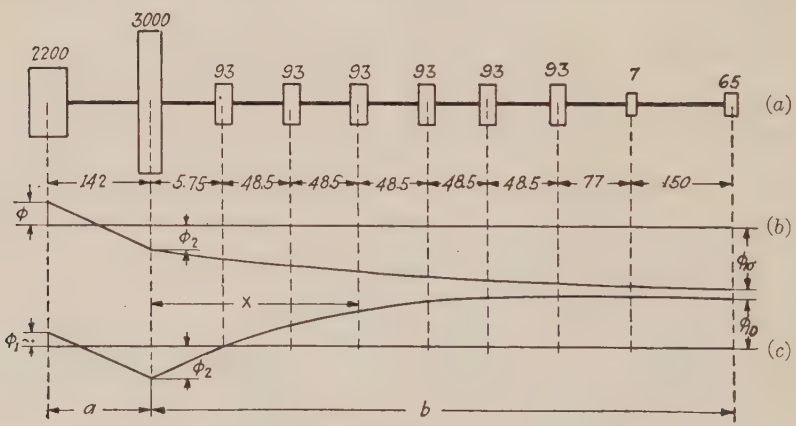


FIG. 76.

section (see p. 157 below) with a torsional rigidity $C = 10^{10}$ kg. $\times$ cm^2. Due to the fact that the masses of the generator and of the flywheel are much larger than the remaining masses a good approximation for the

* This example is discussed in the book by Holzer mentioned below (see p. 22). Kilogram and centimeter are taken as units.

frequency of the lowest type of vibration can be obtained by replacing all the small masses by one mass having a moment of inertia, $I_3 = 93 \times 6 + 7 + 6.5 \approx 572$ and located at the distance $57.5 + 2.5 \times 48.5 \approx 179$ centimeters from the flywheel. Reducing in this manner the given system to three masses only the frequencies can be easily calculated from eq. 77 and we obtain $p_1^2 = 49000$ and $p_2^2 = 123000$. The exact solution for the same problem gives $p_1^2 = 49840$ and $p_2^2 = 141000$. It is seen that a good approximation is obtained for the fundamental type of vibration. In order to get a still better approximation Rayleigh's method can be used (see paragraph 14).

Rayleigh's Method.—Let a and b denote the distances of the flywheel from the ends of the shaft and assume that the shapes of the two principal modes of vibrations are such as shown in Fig. 76 (b) and (c) and that the part b of the deflection curve can be replaced by a parabola so that the angle of twist φ for any section distant x from the flywheel is given by the equation

$$\varphi = \varphi_2 + \frac{(\varphi_{10} - \varphi_2)(2b - x)x}{b^2}. \tag{r}$$

It is easy to see that for $x = 0$ and $x = b$ the angle φ in the above equation assumes the values φ_2 and φ_{10} respectively. By eq. (r) and the first of eqs. (d) we have:

$$\varphi_2 = \varphi_1\left(1 - \frac{I_1 p^2}{k_1}\right). \tag{s}$$

The angles of rotation of all other masses can be represented as functions of φ_1 and φ_{10} and these latter two angles can be considered as the generalized coordinates of the given system.

Then the potential energy of the system is

$$V = \frac{(\varphi_1 - \varphi_2)^2 C}{2a} + \frac{1}{2} C \int_0^b \left(\frac{d\varphi}{dx}\right)^2 dx = \frac{C}{2}\left\{\frac{\varphi_1^2\gamma^2}{a} + \frac{4}{3}\frac{\{\varphi_{10} - \varphi_1(1 - \gamma)\}^2}{b}\right\}, \tag{t}$$

in which

$$\gamma = \frac{I_1 p^2}{k_1}. \tag{u}$$

The kinetic energy of the system will be

$$T = \frac{1}{2}\sum_{k=1}^{k=10} \frac{I_k \dot{\varphi}_k^2}{2},$$

or by using the eqs. (r) and (s) and letting x_k = the distance from the flywheel to any rotating mass k and $\alpha_k = \dfrac{(2b - x_k)x_k}{b^2}$

$$T = \frac{1}{2} I_1 \dot{\varphi}_1^2 + \frac{1}{2}\sum_{k=2}^{k=10} I_k\{\dot{\varphi}_{10}\alpha_k + \dot{\varphi}_1(1 - \gamma)(1 - \alpha_k)\}^2. \tag{v}$$

Substituting (v) and (t) in Lagrange's equations (71) and putting, as before,

$$\varphi_1 = \lambda_1 \cos(pt + \beta); \qquad \varphi_{10} = \lambda_{10} \cos(pt + \beta),$$

the following two equations will be obtained:

$$\lambda_1\left\{\gamma - \frac{4}{3}\frac{a}{b}(1-\gamma) + \frac{\gamma(1-\gamma)}{I_1}\sum_{k=2}^{k=10} I_k(1-\alpha_k)^2\right\} + \lambda_{10}\left\{\frac{4}{3}\frac{a}{b} + \frac{\gamma}{I_1}\sum_{k=2}^{k=10} I_k\alpha_k(1-\alpha_k)\right\} = 0,$$

$$\lambda_1\left\{\frac{4}{3}\frac{a}{b}(1-\gamma) + \frac{\gamma(1-\gamma)}{I_1}\sum_{k=2}^{k=10} I_k\alpha_k(1-\alpha_k)\right\} + \lambda_{10}\left\{-\frac{4}{3}\frac{a}{b} + \frac{\gamma}{I_1}\sum_{k=2}^{k=10} I_k\alpha_k^2\right\} = 0.$$

By equating the determinant of these equations to zero the frequency equation will be obtained, the two roots of which will give us the frequencies of the two modes of vibration shown in Fig. 76. All necessary calculations are given in the table on p. 149.

Then from the frequency equation the smaller root will be

$$\gamma = 1.563$$

and from (u) we obtain

$$p^2 = 50000.$$

The error of this approximate solution as compared with the exact solution given above is only 1/6%.

The second root of the frequency equation gives the frequency of the second mode of vibration with an accuracy of 4.5%. It should be noted that in using this approximate method the effect of the mass of the shaft on the frequency of the system can easily be calculated.*

In the previous discussion the free torsional vibration of shafts was considered and the frequencies of the natural vibrations were determined. If variable torque moments are acting on the shaft a forced vibration in addition to the free vibration considered above will be produced. The period of these forced vibrations will be equal to the period of the variable torque. The amplitude of the forced vibrations will depend not only on the value of the variable torque but also upon its frequency. If this frequency approaches one of the natural frequencies of the shaft, calculated above, a resonance condition takes place and the amplitude of the forced vibration becomes very large, provided there is not sufficient damping. In the following an approximate method for calculating the frequencies of the natural modes of the vibration and the amplitude of the forced vibrations will be discussed.

Calculation of Frequencies by Successive Approximations.—The frequencies of the various modes of torsional vibration can be calculated from eqs. (d) also by successive approximations. For this purpose these equations must be written in the form:

$$\lambda_2 = \lambda_1 - \frac{I_1 p^2}{k_1}\lambda_1, \qquad (w)$$

* See writer's paper in the Bulletin of the Polytechnical Institute in S. Petersburg (1905).

k	I_k	x_k	$\frac{x_k}{b}$	$2-\frac{x_k}{b}$	α_k	$1-\alpha_k$	α_k^2	$I_k\alpha_k^2$	$(1-\alpha_k)^2$	$I_k(1-\alpha_k)^2$	$\alpha_k(1-\alpha_k)$	$I_k\alpha_k(1-\alpha_k)$
1	2200	—	—	—	—	—	—	—	—	—	—	—
2	3000	0	0	2	0	1	0	0	1	3000	0	0
3	93	57.5	.1091	1.8909	.2062	.7938	.0425		.6301		.1637	
4	93	106	.2011	1.7989	.3618	.6382	.1309		.4073		.2309	
5	93	154.5	.2932	1.7068	.5004	.4996	.2504		.2496		.2500	
6	93	203	.3852	1.6148	.6220	.3780	.3869	186.2	.1429	133.8	.2351	121.2
7	93	251.5	.4772	1.5228	.7268	.2732	.5282		.0746		.1986	
8	93	300	.5693	1.4307	.8145	.1855	.6634		.0344		.1511	
9	7	377	.7154	1.2846	.9190	.0810	.8446	5.91	.0065		.0744	.52
10	6.5	527	1.0000	1.0000	1.0000	0	1.0000	6.50	0	0	0	0

$$\sum_{k=2}^{k=10} I_k\alpha_k^2 = 198.6; \quad \sum_{k=2}^{k=10} I_k(1-\alpha_k)^2 = 3133.8; \quad \sum_{k=2}^{k=10} I_k\alpha_k(1-\alpha_k) = 121.7.$$

$$\lambda_3 = \lambda_2 - \frac{p^2}{k_2}(I_1\lambda_1 + I_2\lambda_2), \tag{y}$$

$$\lambda_4 = \lambda_3 - \frac{p^2}{k_3}(I_1\lambda_1 + I_2\lambda_2 + I_3\lambda_3), \tag{z}$$

. . . . . .

. . . . . .

Making now a rough estimate of the value p^2 and taking an arbitrary value for λ_1, the angular deflection of the first disc, the corresponding value of λ_2 will be found from eq. (w). Then, from eq. (y) λ_3 will be found; λ_4 from eq. (z) and so on. If the magnitude of p^2 had been chosen correctly, the equation

$$I_1\lambda_1 p^2 + I_2\lambda_2 p^2 + \cdots I_n\lambda_n p^2 = 0,$$

representing the sum of the eqs. (d), would be satisfied. Otherwise the angles λ_2, λ_3, $\cdots$ would have to be calculated again with a new estimate for p^2.* It is convenient to put the results of these calculations in tabular form. As an example, the calculations for a Diesel installation, shown in figure 77, are given in the tables on p. 151.†

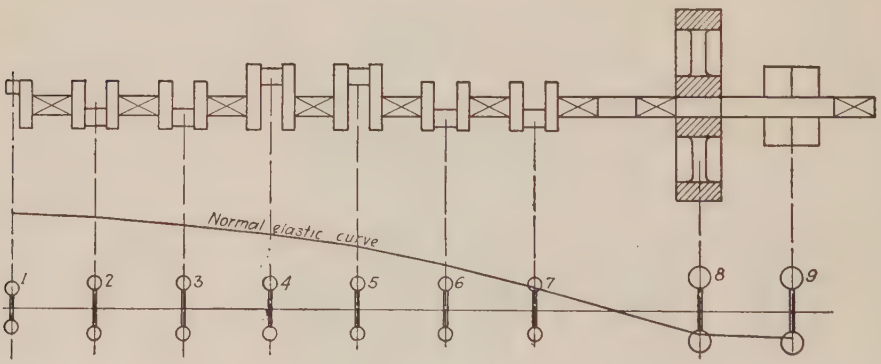


Fig. 77.

Column 1 of the tables gives the moments of inertia of the masses, inch, pound and second being taken as units. Column 3 begins with an arbitrary value of the angle of rotation of the first mass. This angle is

* Several examples of this calculation may be found in the book by H. Holzer. 'Die Berechnung der Drehschwingungen," 1921, Berlin J. Springer. See also F. M., Lewis, loc. cit., and Max Tolle, "Regelung der Kraftmaschinen," 3d Ed., 1921.

† These calculations were taken from the paper of F. M. Lewis, mentioned above.

Table for $p = 96{,}2$; $p^2 = 9250$

Mass No.	1	2	3	4	5	6	7
	I	Ip^2	λ	$Ip^2\lambda$	$\Sigma Ip^2\lambda$	k	$\frac{1}{k}\Sigma Ip^2\lambda$
1	708	$6{,}55\times10^6$	1	$6{,}55\times10^6$	$6{,}55\times10^6$	2070×10^6	.0031
2	3920	$36{,}25\times10^6$	.9969	$36{,}1\times10^6$	$42{,}65\times10^6$	730×10^6	.0585
3	3920	$36{,}25\times10^6$	.9383	$34{,}0\times10^6$	$76{,}65\times10^6$	730×10^6	.1050
4	3920	$36{,}25\times10^6$	.8333	$30{,}2\times10^6$	$106{,}85\times10^6$	730×10^6	.1462
5	3920	$36{,}25\times10^6$	.6871	$24{,}9\times10^6$	$131{,}75\times10^6$	730×10^6	.1803
6	3920	$36{,}25\times10^6$	.5068	$18{,}38\times10^6$	$150{,}1\times10^6$	730×10^6	.2060
7	3920	$36{,}25\times10^6$	.3008	$10{,}90\times10^6$	$161{,}0\times10^6$	402×10^6	.4010
8	139800	1293×10^6	−.1002	$-130{,}0\times10^6$	$31{,}0\times10^6$	1334×10^6	.0233
9	26400	244×10^6	−.1235	$-30{,}2\times10^6$	+800,000		

Table for $p = 96{,}8$; $p^2 = 9380$

Mass No.	1	2	3	4	5	6	7
	I	Ip^2	λ	$Ip^2\lambda$	$\Sigma Ip^2\lambda$	k	$\frac{1}{k}\Sigma Ip^2\lambda$
1	708	$6{,}65\times10^6$	1	$6{,}65\times10^6$	$6{,}65\times10^6$	2070×10^6	.0032
2	3920	$36{,}8\times10^6$	.9968	$36{,}7\times10^6$	$43{,}35\times10^6$	730×10^6	.0594
3	3920	$36{,}8\times10^6$	.9374	$34{,}5\times10^6$	$77{,}85\times10^6$	730×10^6	.1069
4	3920	$36{,}8\times10^6$	.8305	$30{,}6\times10^6$	$108{,}45\times10^6$	730×10^6	.1487
5	3920	$36{,}8\times10^6$	.6818	$25{,}1\times10^6$	$133{,}55\times10^6$	730×10^6	.1830
6	3920	$36{,}8\times10^6$	.4988	$18{,}36\times10^6$	$151{,}91\times10^6$	730×10^6	.2080
7	3920	$36{,}8\times10^6$	.2908	$10{,}70\times10^6$	$162{,}61\times10^6$	402×10^6	.4040
8	139800	1312×10^6	−.1132	$-148{,}5\times10^6$	$14{,}11\times10^6$	1334×10^6	.0106
9	26400	248×10^6	−.1238	$-30{,}70\times10^6$	$-16{,}59\times10^6$		

taken equal to 1. Column 4 gives the moments of the inertia forces of the consecutive masses and column 5 the total torque moment of the inertia forces of all masses to the left of the cross section considered. Dividing the torque by the spring constants given in column 6, we obtain the angles of twist for consecutive portions of the shaft. These are given in column 7. The last number in column 5 represents the sum of the moments of the inertia forces of all the masses. This sum must be equal to zero in the case of free vibration. By taking $p = 96.2$ in the first table, the last value in column 5 becomes positive. For $p = 96.8$ the corresponding value is negative. This shows that the exact value of p lies between the above two values and the correct values in columns 3 and 5 will be obtained by interpolation. By using the values in column 3, the elastic curve representing the *mode of vibration* can be

constructed as shown in figure 77. Column 5 gives the corresponding torque moments for each portion of the shaft when the amplitude of the first mass is 1 radian. If this amplitude has any other value λ_1, the amplitudes and the torque moments of the other masses may be obtained by multiplying the values in columns 3 and 5 by λ_1.

*Calculations of forced vibrations.**—If there is no friction and a simple harmonic torque $M \sin mt$ is applied to one of the rotating masses, forced vibration of the period $\tau = (2\pi/m)$ will be produced and the vibration of each mass will be of the form $\lambda \sin mt$. The amplitudes of forced vibration may be determined by preparing a table like the one used above in the case of free vibration. Assuming that the amplitude of the first mass is x, the torque moment in each interval may be calculated in the same manner as before; it is only necessary to add the maximum M of the external torque at its point of application to the moment of the inertia forces of the rotating masses. The last term of column 5 will be of the form $ax + b$ and a linear equation for determining x will be obtained by equating this term to zero, since it represents the moment beyond the last mass.

When the period of the external harmonic torque coincides with the period of one of the natural modes of vibration of the system, a condition of resonance takes place. This mode of vibration becomes very pronounced and the damping forces must be taken into consideration in order to obtain the actual value of the amplitude of vibration. Assuming that the damping force is proportional to the velocity and neglecting the effect of this force on the *mode of vibration*, i.e., assuming that the ratios between the amplitudes of the steady forced vibration of the rotating masses are the same as for the corresponding type of free vibration, the approximate values of the amplitudes of forced vibration may be calculated as follows: Let $\varphi_m = \lambda_m \sin pt$ be the angle of rotation during vibration of the mass on which damping is acting. Then the resisting moment of the damping forces will be

$$-\alpha \frac{d\varphi_m}{dt} = -\alpha\lambda_m p \cos pt,$$

where α is a constant depending upon the damping condition. The phase difference between the torque moment which produces the forced

* The approximate method of calculating forced vibration with damping has been developed by H. Wydler in the book: "Drehschwingungen in Kolbenmaschinenanlagen." Berlin, 1922. See also F. M. Lewis, loc. cit., p. 138, and John F. Fox, Some Experiences with Torsional Vibration Problems in Diesel Engine Installations, Journal Amer. Soc. of Naval Engineers 1926.

vibration and the displacement must be 90 degrees for resonance. Taking this moment in the form $M \cos pt$ and assuming $\varphi_n = \lambda_n \sin pt$ for the angle of rotation of the mass on which the torque is acting, the amplitude of the forced vibration will be found from the condition that in the steady state of forced vibration the work done by the harmonic torque during one oscillation must be equal to the energy absorbed at the damping point. In this manner we obtain

$$\int_0^{\frac{2\pi}{p}} \alpha \frac{d\varphi_m}{dt} \frac{d\varphi_m}{dt} dt = \int_0^{\frac{2\pi}{p}} M \cos pt \frac{d\varphi_n}{dt} dt,$$

or substituting

$$\varphi_m = \lambda_m \sin pt; \qquad \varphi_n = \lambda_n \sin pt,$$

we obtain

$$\lambda_m = \frac{M}{\alpha p} \frac{\lambda_n}{\lambda_m}, \tag{a}^1$$

and the amplitude of vibration for the first mass will be

$$\lambda_1 = \frac{M}{\alpha p} \frac{\lambda_n}{\lambda_m} \frac{\lambda_1}{\lambda_m}. \tag{b}^1$$

Knowing the damping constant α and taking the ratios λ_n/λ_m and λ_1/λ_m from the *normal elastic curve* (see fig. 77) the amplitudes of forced vibration may be calculated for the case of a simple harmonic torque with damping applied at certain section of the shaft.

If several simple harmonic torques are acting on the shaft, the resultant amplitude λ_1, of the first mass, may be obtained from the equation $(b)^1$ above by the principle of superposition. It will be equal to

$$\lambda_1 = \frac{1}{\alpha p} \frac{\lambda_1}{\lambda_m^2} \sum M\lambda_n, \tag{c}^1$$

where the summation sign indicates the vector sum, each torque moment being taken with the corresponding phase.

In actual cases the external torque moment is usually of a more complicated nature. In the case of a Diesel engine, for instance, the *turning effort* produced by a single cylinder depends on the position of the crank, on the gas pressure and on inertia forces. The *turning effort curve* of each cylinder may be constructed from the corresponding gas pressure diagram, taking into account the inertia forces of the reciprocating

masses. In analyzing forced vibrations this curve must be represented by a trigonometrical series *

$$f(\varphi) = a_0 + a_1 \cos \varphi + a_2 \cos 2\varphi + \cdots + b_1 \sin \varphi + b_2 \sin 2\varphi + \cdots, \qquad (d)^1$$

in which $\varphi = 2\pi$ represents the period of the curve. This period is equal to one revolution of the crank shaft in a two cycle engine and to two revolutions in a four cycle engine. The condition of resonance occurs and a critical speed will be obtained each time when the frequency of one of the terms of the series $(d)^1$ coincides with the frequency of one of the natural modes of vibration of the shaft. For a single cylinder in a two cycle engine there will be obtained in this manner critical speeds of the order $1, 2, 3, \cdots$, where the index indicates the number of vibration cycles per revolution of the crank shaft. In the case of a four-cycle engine, we may have critical speeds of the order $\frac{1}{2}, 1, 1\frac{1}{2}, \cdots$; i.e., of every integral order and half order. There will be a succession of such critical speeds for each mode of natural vibration. The amplitude of a forced vibration of a given type produced by a single cylinder may be calculated as has been explained before. In order to obtain the summarized effect of all cylinders, it will be necessary to use the principle of superposition, taking the turning effort of each cylinder at the corresponding phase. In particular cases, when the number of vibrations per revolution is equal to, or a multiple of the number of firing impulses (a major critical speed) the phase difference is zero and the vibrations produced by the separate cylinders will be simply added together. Several examples of the calculation of amplitudes of forced vibration are to be found in the papers by H. Wydler and F. M. Lewis mentioned above. They contain also data on the amount of damping in such parts as the marine propeller, the generator, and the cylinders as well as data on the losses due to internal friction.† The application in particular cases of the described approximate method gives satisfactory accuracy in computing the amplitude of forced vibration and the corresponding maximum stress.

Geared Systems.—The method developed above for calculating the frequencies of torsional vibration of shafts can be used also in the case of geared systems such as shown in Fig. 78(*a*). Let

* Examples of such an analysis may be found in the papers by H. Wydler, loc. cit., p. 152, and F. M. Lewis, loc. cit., p. 138.

† Bibliography on this subject and some new data on internal friction may be found in the book by E. Lehr, Die Abkürzungsverfahren zur Ermittelung der Schwingungsfestigkeit, Stuttgart, dissertation, 1925.

I_1, I_3 = moments of inertia of rotating masses.
φ_1, φ_3 = the corresponding angles of rotation.
i_2', i_2'' = moments of inertia of gears.
n = gear ratio.
$\varphi_2, \quad -n\varphi_2$ = angles of rotation of gears.
k_1, k_2 = spring constants of shafts.

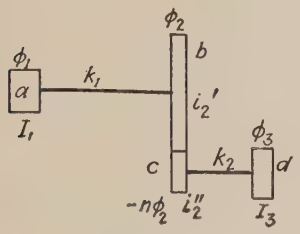


FIG. 78(a)

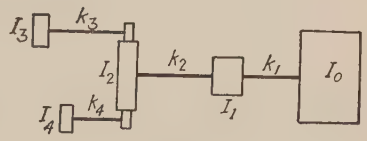


FIG. 78b.

Then the kinetic energy of the system will be

$$T = \frac{I_1\dot{\varphi}_1^2}{2} + \frac{i_2'\dot{\varphi}_2^2}{2} + \frac{i_2''(n\dot{\varphi}_2)^2}{2} + \frac{I_3\dot{\varphi}_3^2}{2}. \qquad (e)'$$

The potential energy of the system is

$$V = \tfrac{1}{2}k_1(\varphi_1 - \varphi_2)^2 + \tfrac{1}{2}k_2(\varphi_3 + n\varphi_2)^2. \qquad (f)'$$

Letting

$$\begin{aligned} i_2' + n^2 i_2'' &= I_2; \qquad & n^2 I_3 &= I_3', \\ \varphi_3 &= -n\varphi_3'; \qquad & k_2 n^2 &= k_2'. \end{aligned} \qquad (g)'$$

The equations (a) and (b) become

$$T = \frac{I_1\dot{\varphi}_1^2}{2} + \frac{I_2\dot{\varphi}_2^2}{2} + \frac{I_3'(\dot{\varphi}_3')^2}{2},$$

$$V = \tfrac{1}{2}k_1(\varphi_1 - \varphi_2)^2 + \tfrac{1}{2}k_2'(\varphi_2 - \varphi_3')^2.$$

These expressions have the same form as the expressions (a) and (b) obtained above for the torsional vibration of a shaft. It can be concluded from this that the differential equations of vibration of the geared system shown in Fig. 78(a) will be the same as those of a shaft with discs provided the notations shown in eqs. (g)′ are used. This conclusion can be expanded also to the case of a geared system with more than two shafts.*

Another arrangement of a geared system is shown in Fig. 78b, in

* Such systems are considered in the paper by T. H. Smith, "Nodal Arrangements of Geared Drives," Engineering, 1922, p. 438 and 467.

which I_0, I_1, I_2, $\cdots$ are the moments of inertia of the rotating masses; k_1, k_2, $\cdots$ torsional rigidities of the shafts. Let, n = gear ratio, φ_0, φ_1, φ_2, $\cdots$ = angles of rotation of discs I_0, I_1 I_2 $\cdots$. If I_0 is very large in comparison with the other moments of inertia we can take $\varphi_0 = 0$, then the kinetic and the potential energy of the system will be

$$T = \tfrac{1}{2}(I_1\dot{\varphi}_1^2 + I_2\dot{\varphi}_2^2 + I_3\dot{\varphi}_3^2 + I_4\dot{\varphi}_4^2),$$
$$V = \tfrac{1}{2}\{k_1\varphi_1^2 + k_2(\varphi_2 - \varphi_1)^2 + k_3(\varphi_3 + n\varphi_2)^2 + k_4(\varphi_4 + n\varphi_2)^2\},$$

and Lagrange's differential equations of motion become

$$\begin{aligned} I_1\ddot{\varphi}_1 + k_1\varphi_1 - k_2(\varphi_2 - \varphi_1) &= 0, \\ I_2\ddot{\varphi}_2 + k_2(\varphi_2 - \varphi_1) + nk_3(\varphi_3 + n\varphi_2) + nk_4(\varphi_4 + n\varphi_2) &= 0, \\ I_3\ddot{\varphi}_3 + k_3(\varphi_3 + n\varphi_2) &= 0, \\ I_4\ddot{\varphi}_4 + k_4(\varphi_4 + n\varphi_2) &= 0, \end{aligned}$$

from which the frequency equation can be obtained in the same manner as before and the frequencies will then be represented by the roots of this equation.

Torsional Vibrations in Crank Shafts.—In the case of crank shafts the problem of the torsional rigidity of one crank (Fig. 79) must be considered

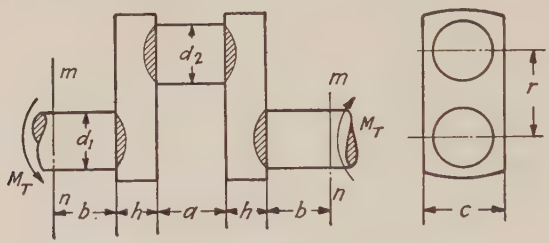


FIG. 79.

first. This rigidity depends on the conditions of constraint at the bearings. Assuming that the clearances in the bearings are such that free displacements of the cross sections m–n and m–n during twist are possible, the angle of twist produced by a torque moment M_T can be easily obtained. This angle consists of three parts; (a) twist of the journals, (b) twist of the crankpin and (c) bending of the web.

Let $C_1 = \dfrac{\pi d_1^4 G}{32}$ = the torsional rigidity of the journal,

$C_2 = \dfrac{\pi d_2^4 G}{32}$ = the torsional rigidity of the crankpin,

$B = \dfrac{hc^3}{12}E$ = the flexural rigidity of the web.

In order to take into account local deformations in the web in the regions shaded in the figure, due to twist, the lengths of the journal and of the pin are taken equal to $2b_1 = 2b + .9h$ and $a_1 = a + .9h$, respectively.* The angle of twist θ of the crank produced by a torque moment M_T will then be

$$\theta = \frac{2b_1 M_T}{C_1} + \frac{a_1 M_T}{C_2} + \frac{2r M_T}{B}.$$

In calculating the torsional vibrations of a crankshaft every crank can be replaced by an equivalent shaft of uniform cross section of a torsional rigidity C. The length of the equivalent shaft will be found from

$$\frac{M_T l}{C} = \theta,$$

in which θ is the angle of twist calculated above.

Then the length of equivalent shaft will be,

$$l = C\left(\frac{2b_1}{C_1} + \frac{a_1}{C_2} + \frac{2r}{B}\right). \tag{78}$$

Another extreme case will be obtained on the assumption that the constraint at the bearings is complete, corresponding to no clearances. In this case the length l of the *equivalent shaft* will be found from the equation,*

$$l = C\left\{\frac{2b_1}{C_1} + \frac{a_1}{C_2}\left(1 - \frac{r}{k}\right) + \frac{2r}{B}\left(1 - \frac{r}{2k}\right)\right\}, \tag{79}$$

in which

$$k = \frac{\dfrac{r(a+h)^2}{4C_3} + \dfrac{ar^2}{2C_2} + \dfrac{a^3}{24B_1} + \dfrac{r^3}{3B} + \dfrac{1.2}{G}\left(\dfrac{a}{2F} + \dfrac{r}{F_1}\right)}{\dfrac{ar}{2C_2} + \dfrac{r^2}{2B}}, \tag{80}$$

in which again

$C_3 = \dfrac{c^3h^3G}{3.6(c^2 + h^2)}$ = the torsional rigidity of the web as a bar of rectangular cross section with sides h and c,

$B_1 = \dfrac{\pi d_2{}^4 E}{64}$ = the flexural rigidity of the crankpin,

* Such an assumption is in good agreement with experiments made; see a paper by Dr. Seelmann, V.D.I. Vol. 69 (1925), p. 601, and F. Sass, Maschinenbau, Vol. 4, 1925, p. 1223. See also F. M. Lewis, loc. cit., p. 138.

† A detailed consideration of the twist of a crankshaft is given by the writer in Trans. Am. Soc. Mech. Eng., Vol. 44 (1922), p. 653. See also "Applied Elasticity," p. 188.

F, F_1 = the cross sectional areas of the pin and of the web, respectively.

By taking $a_1 = 2b_1$ and $C_1 = C_2$ the complete constraint as it is seen from eqs. (78) and (79) diminishes the equivalent length of shaft in the ratio $1 : \{1 - (r/2k)\}$. In actual conditions the length of the equivalent shaft will have an intermediate value between the two extreme cases considered above.

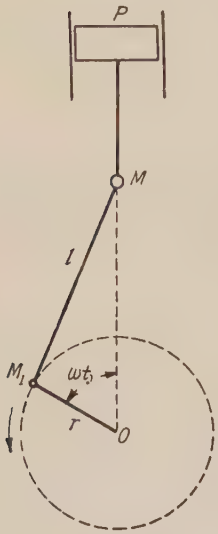


FIG. 80.

Another question to be decided in considering torsional vibration of crankshafts is the calculation of the inertia of the moving masses. Let us assume that the mass m of the connecting rod is replaced in the usual way * by two masses $m_1 = (I/l^2)$ at the crankpin and $m_2 = m - (I/l^2)$ at the cross head, where I denotes the moment of inertia of the connecting rod about the center of cross head. All other moving masses also can be replaced by masses concentrated in the same two points so that finally only two masses M and M_1 must be taken into consideration (Fig. 80). Let ω = constant angular velocity,

ωt = the angle of the crank measured from the dead position as shown in Fig. 80.

Then the velocity of the mass M_1 is equal to ωr and the velocity of the mass M, as shown in paragraph 12 (see p. 50), is equal to

$$\omega r \sin \omega t + \frac{r^2\omega}{2l} \sin 2\omega t.$$

The kinetic energy of the moving masses of one crank will be

$$T = \tfrac{1}{2}M_1\omega^2r^2 + \tfrac{1}{2}M\omega^2r^2\left(\sin \omega t + \frac{r}{2l}\sin 2\omega t\right)^2.$$

The average value of T during one revolution is

$$T_0 = \frac{1}{2\pi}\int_0^{2\pi} T d(\omega t) = \frac{1}{2}\left\{M_1 + \frac{1}{2}M\left(1 + \frac{r^2}{4l^2}\right)\right\}\omega^2r^2.$$

By using this average value, the inertia of the moving parts connected with one crank can be replaced by the inertia of an equivalent disc

* See, for instance, "Regelung der Kraftmaschinen," by Max Tolle, 3d Ed. (1921), p. 116.

having a moment of inertia

$$I = \left\{ M_1 + \frac{1}{2} M \left(1 + \frac{r^2}{4l^2} \right) \right\} r^2.$$

By replacing all cranks by *equivalent lengths* of shaft and all moving masses by *equivalent discs* the problem on the vibration of crankshafts will be reduced to that of the torsional vibration of shafts and the critical speeds can be calculated as has been shown before. It should be noted that such a method of investigating the vibration must be considered only as a rough approximation. The real problem is much more complicated and in the simplest case of only one crank with a flywheel it reduces to a problem in torsional vibrations of a shaft with two discs, one of which has a variable moment of inertia. More detailed investigations show * that in such a system "forced vibrations" do not arise only from the pressure of the expanding gases on the piston. They are also produced by the incomplete balance of the reciprocating parts. Practically all the phenomena associated with dangerous critical speeds would appear if the fuel were cut off and the engine made to run without resistance at the requisite speed.

The positions of the critical speeds in such systems are approximately those found by the usual method, i.e., by replacing the moving masses by *equivalent discs.*

32. Lateral Vibrations of Shafts on Many Supports.—*General.*—In our previous discussion (paragraph 15) the simplest case of a shaft on two supports was considered and it was then shown that the critical speed of rotation of a shaft is that speed at which the number of revolutions per second is equal to the frequency of its natural lateral vibrations. In practice, however, cases of shafts on *many* supports are encountered and consideration will now be given to the various methods which may be employed for calculating the frequencies of the natural modes of lateral vibration of such shafts.†

Analytical Method.—This method can be applied without difficulty in the case of a shaft of uniform cross section carrying several discs.

Let us consider first the simple example of a shaft on three supports carrying two discs (Fig. 81) the weights of which are w_1 and w_2. The statical deflections of the shaft under these loads can be represented by

* See paper by G. R. Goldsbrough, "Torsional Vibration in Reciprocating Engine Shafts," Proc. of the Royal Society, Vol. 109 (1925), p. 99.

† This subject is discussed in detail by A. Stodola, "Dampf- und Gasturbinen," 6th Ed. Berlin 1924.

the equations,

$$y_1 = a_{11}w_1 + a_{12}w_2, \qquad (a)$$

$$y_2 = a_{21}w_1 + a_{22}w_2, \qquad (b)$$

the constants a_{11}, a_{12}, a_{21} and a_{22} of which can be calculated in the following manner. Remove the intermediate support C and consider the deflections

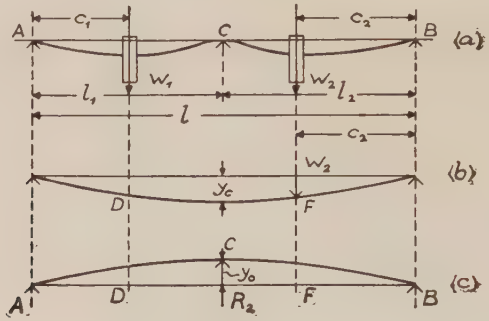


FIG. 81.

produced by load w_2 alone (Fig. 81, b); then the equation of the deflection curve for the left part of the shaft will be

$$y = \frac{w_2c_2}{6lEI}(-x^3 + l^2x - c_2^2x) \qquad (c)$$

and the deflection at the point C becomes:

$$y_c = \frac{w_2c_2}{6lEI}(-l_1^3 + l^2l_1 - c_2^2l_1).$$

Now determine the reaction R_2 in such a manner as to reduce this deflection to zero (Fig. 81, c). Applying eq. (c) for calculating the deflection under R_2 and putting this deflection equal to y_c, obtained above, we have,

$$\frac{w_2c_2}{6lEI}(-l_1^3 + l^2l_1 - c_2^2l_1) = \frac{R_2l_2}{6lEI}(-l_1^3 + l^2l_1 - l_2^2l_1),$$

from which

$$R_2 = \frac{w_2c_2(l^2 - l_1^2 - c_2^2)}{2l_1l_2^2}.$$

In the same manner the reaction R_1 produced by the load w_1 can be calculated and the complete reaction $R = R_1 + R_2$ at the middle support will be obtained. Now, by using eq. (c) the deflection y_1 produced by

the loads w_1, w_2 and the reaction R can be represented in the form (a) in which

$$a_{11} = \frac{1}{12ll_1^2EI}\{4l_1^2(l-c_1)^2c_1^2 - c_1(-c_1^3+l^2c_1-l_2^2c_1)(l^2-l_2^2-c_1^2)\},$$
$$a_{12} = \frac{1}{12ll_1l_2EI}\{2l_1l_2c_1c_2(l^2-c_1^2-c_2^2) - c_2c_1(l^2-l_2^2-c_1^2)(l^2-l_1^2-c_2^2)\}. \qquad (d)$$

Interchanging l_2 and l_1 and c_2 and c_1 in the above equations, the constants a_{21} and a_{22} of eq. (b) will be obtained and it will be seen that $a_{12} = a_{21}$, i.e., that a load put at the location D (Fig. 81, c) produces at F the same deflection as a load of the same magnitude at F produces at D. Such a result should be expected on the basis of the *reciprocal theorem.*

Consider now the vibration of the loads w_1 and w_2 about their position of equilibrium, found above, and in the plane of the figure. Let y_1 and y_2 now denote the variable displacements of w_1 and w_2 from their positions of equilibrium during vibration. Then, neglecting the mass of the shaft, the kinetic energy of the system will be

$$T = \frac{w_1}{2g}(\dot{y}_1)^2 + \frac{w_2}{2g}(\dot{y}_2)^2. \qquad (e)$$

In calculating the increase in potential energy of the system due to displacement from the position of equilibrium equations (a) and (b) for static deflections will be used. Letting, for simplicity, $a_{11} = a$, $a_{12} = a_{21} = b$, $a_{22} = c$,* we obtain from the above equations (a) and (b) the following forces necessary to produce the deflections y_1 and y_2.

$$P_1 = \frac{cy_1 - by_2}{ac - b^2}; \qquad P_2 = \frac{ay_2 - by_1}{ac - b^2};$$

and †

$$V = \frac{P_1y_1}{2} + \frac{P_2y_2}{2} = \frac{1}{2(ac - b^2)}(cy_1^2 - 2by_1y_2 + ay_2^2). \qquad (f)$$

Substituting (e) and (f) in Lagrange's equations (71) we obtain the following differential equations for the free lateral vibration of the

* The constants a, b and c can be calculated for any particular case by using eqs. (d).

† Due to the fact that the weights w_1 and w_2 are always in balance with the initial bending stresses due to the static deflection of the shaft, the expression (f) for the potential energy will contain only terms due to the *change* in bending of the shaft (see p. 52).

shaft

$$\begin{aligned} \frac{w_1}{g}\ddot{y}_1 + \frac{c}{ac - b^2}y_1 - \frac{b}{ac - b^2}y_2 &= 0, \\ \frac{w_2}{g}\ddot{y}_2 - \frac{b}{ac - b^2}y_1 + \frac{a}{ac - b^2}y_2 &= 0. \end{aligned} \tag{g}$$

Assuming that the shaft performs one of the natural modes of vibration and substituting in eqs. (g):

$$y_1 = \lambda_1 \cos (pt - \alpha); \qquad y_2 = \lambda_2 \cos (pt - \alpha),$$

we obtain

$$\begin{aligned} \lambda_1\left(\frac{c}{ac - b^2} - \frac{w_1}{g}p^2\right) - \frac{b}{ac - b^2}\lambda_2 &= 0, \\ -\frac{b}{ac - b^2}\lambda_1 + \lambda_2\left(\frac{a}{ac - b^2} - \frac{w_2}{g}p^2\right) &= 0. \end{aligned} \tag{h}$$

By putting the determinant of these equations equal to zero the following *frequency equation* will be obtained

$$\left(\frac{c}{ac - b^2} - \frac{w_1}{g}p^2\right)\left(\frac{a}{ac - b^2} - \frac{w_2}{g}p^2\right) - \frac{b^2}{(ac - b^2)^2} = 0, \tag{k}$$

from which

$$p^2 = \frac{g}{2(ac - b^2)}\left\{\frac{c}{w_1} + \frac{a}{w_2} \pm \sqrt{\left(\frac{c}{w_1} + \frac{a}{w_2}\right)^2 - \frac{4(ac - b^2)}{w_1 w_2}}\right\}. \tag{81}$$

In this manner two positive roots for p^2, corresponding to the two principal modes of vibration of the shaft are obtained. Substituting these two roots in one of the eqs. (h) two different values for the ratio λ_1/λ_2 will be obtained. For the larger value of p^2 the ratio λ_1/λ_2 becomes positive, i.e., both discs during the vibration move simultaneously in the same direction and the mode of vibration is as shown in Fig. 82, a. If the smaller root of p^2 be substituted in eq. (h) the ratio λ_1/λ_2 becomes negative and the corresponding mode of vibration will be as shown in Fig. 82, b. Take, for instance, the particular case when (see Fig. 81) $w_1 = w_2$; $l_1 = l_2 = (l/2)$ and $c_1 = c_2 = (l/4)$. Substituting in eqs. (d) and using the conditions of symmetry, we obtain;

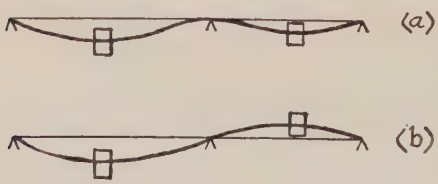


Fig. 82.

$$a = c = \frac{23}{48 \cdot 256} \frac{l^3}{EI} \qquad \text{and} \qquad b = -\frac{9}{48 \cdot 256} \frac{l^3}{EI}.$$

Substituting in eq. 81, we have

$$p_1^2 = \frac{g}{(a-b)w} = \frac{g48EI}{w(l/2)^3}; \qquad p_2^2 = \frac{g768EI}{7w(l/2)^3}.$$

These two frequencies can also be easily derived by substituting in eq. 5 (see p. 2) the statical deflections

$$\delta'_{st} = \frac{w(l/2)^3}{48EI} \qquad \text{and} \qquad \delta''_{st} = \frac{7w(l/2)^3}{768EI}$$

for the cases shown in Fig. 83.

Another method of solution of the problem on the lateral vibrations of shafts consists in the application of D'Alembert's principle. In using this principle the equations of vibration will be written in the same manner as the equations of statics. It is only necessary to add to the loads acting on the shaft the inertia forces. Denoting as before, by y_1 and y_2 the deflections of the shaft from the position of equilibrium under the loads w_1 and w_2, respectively, the inertia forces will be $-(w_1/g)\ddot{y}_1$ and $-(w_2/g)\ddot{y}_2$. These inertia forces must be in equilibrium with the elastic forces due to the additional deflection and two equations equivalent to (a) and (b) can be written down as follows.

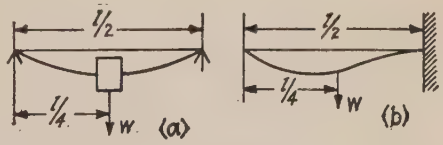


FIG. 83.

$$\begin{aligned} y_1 &= -a\frac{w_1}{g}\ddot{y}_1 - b\frac{w_2}{g}\ddot{y}_2, \\ y_2 &= -b\frac{w_1}{g}\ddot{y}_1 - c\frac{w_2}{g}\ddot{y}_2. \end{aligned} \qquad (l)$$

Assuming, as before,

$$y_1 = \lambda_1 \cos(pt - \alpha); \qquad y_2 = \lambda_2 \cos(pt - \alpha),$$

and substituting in eqs. (l) we obtain,

$$\begin{aligned} \lambda_1\left(1 - \frac{aw_1}{g}p^2\right) - \lambda_2 b\frac{w_2}{g}p^2 &= 0, \\ -\lambda_1 b\frac{w_1}{g}p^2 + \lambda_2\left(1 - \frac{cw_2}{g}p^2\right) &= 0. \end{aligned} \qquad (m)$$

Putting the determinant of these two equations equal to zero, the frequency equation (k), which we had before, will be obtained.

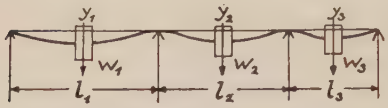


FIG. 84.

The methods developed above for calculating the frequencies of the lateral vibrations can be used also in cases where the number of discs or the number of spans is greater than two. Take, for instance, the case shown in Fig. 84. By using a method analogous to the one employed in the previous example, the statical deflections of the shaft under the discs can be represented in the following form

$$\begin{aligned} y_1 &= a_{11}w_1 + a_{12}w_2 + a_{13}w_3, \\ y_2 &= a_{21}w_1 + a_{22}w_2 + a_{23}w_3, \\ y_3 &= a_{31}w_1 + a_{32}w_2 + a_{33}w_3, \end{aligned} \tag{n}$$

in which a_{11}, a_{12}, $\cdots$ are constants depending on the distances between the supports, the distances of the discs from the supports and on the flexural rigidity of the shaft. From the *reciprocal theorem* it can be concluded at once that $a_{12} = a_{21}$, $a_{13} = a_{31}$ and $a_{23} = a_{32}$. Applying now D'Alembert's principle and denoting by y_1, y_2 and y_3 the displacements of the discs during vibration from the position of equilibrium, the following equations of vibration will be derived from the statical equations (n).

$$\begin{aligned} y_1 &= -a_{11}\frac{w_1}{g}\ddot{y}_1 - a_{12}\frac{w_2}{g}\ddot{y}_2 - a_{13}\frac{w_3}{g}\ddot{y}_3, \\ y_2 &= -a_{21}\frac{w_1}{g}\ddot{y}_1 - a_{22}\frac{w_2}{g}\ddot{y}_2 - a_{23}\frac{w_3}{g}\ddot{y}_3, \\ y_3 &= -a_{31}\frac{w_1}{g}\ddot{y}_1 - a_{32}\frac{w_2}{g}\ddot{y}_2 - a_{33}\frac{w_3}{g}\ddot{y}_3, \end{aligned}$$

from which the *frequency equation*, a cubic in p^2, can be gotten in the usual manner. The three roots of this equation will give the frequencies of the three principal modes of vibration of the system under consideration.*

It should be noted that the frequency equations for the lateral vibrations of shafts can be used also for calculating *critical speeds* of rotation. A critical speed of rotation is a speed at which the centrifugal forces of the rotating masses are sufficiently large to keep the shaft in a

* A graphical method of solution of frequency equations has been developed by C. R. Soderberg, Phil. Mag., Vol. 5, 1928, p. 47.

bent condition (see paragraph 15). Take again the case of two discs (Fig. 81, *a*) and assume that y_1 and y_2 are the deflections, produced by the centrifugal forces * $(w_1/g)\omega^2 y_1$ and $(w_2/g)\omega^2 y_2$ of the rotating discs. Such deflections can exist only if the centrifugal forces satisfy the following conditions of equilibrium [see eqs. (*a*) and (*b*)],

$$\begin{aligned} y_1 &= a_{11}\frac{w_1}{g}\omega^2 y_1 + a_{12}\frac{w_2}{g}\omega^2 y_2, \\ y_2 &= a_{21}\frac{w_1}{g}\omega^2 y_1 + a_{22}\frac{w_2}{g}\omega^2 y_2. \end{aligned} \tag{o}$$

These equations can give for y_1 and y_2 solutions different from zero only in the case when their determinant vanishes. Observing that the equations (*o*) are identical with the equations (*m*) above and equating their determinant to zero, an equation identical with eq. (*k*) will be obtained for calculating the critical speeds of rotation.

Graphical Method.—In the case of shafts of variable cross section or those having many discs the analytical method of calculating the critical speeds, described above, becomes very complicated and recourse should be made to graphical methods. As a simple example, a shaft supported at the ends will now be considered (Fig. 37). Assume some *initial deflection* of the rotating shaft satisfying the end conditions where y_1, y_2, $\cdots$ are the deflections at the discs w_1, w_2, $\cdots$. If ω be the angular velocity then the corresponding centrifugal forces will be $(w_1/g)\omega^2 y_1$, $(w_2/g)\omega^2 y_2$, $\cdots$. Considering these forces as statically applied to the shaft, the corresponding deflection curve can be obtained graphically as was explained in paragraph 15. If our assumption about the shape of the *initial* deflection curve was correct, the deflections y_1', y_2', $\cdots$, as obtained *graphically*, should be proportional to the deflections y_1, y_2, $\cdots$ initially assumed, and the critical speed will be found from the equation

$$\omega_{cr} = \omega\sqrt{\frac{y_1}{y_1'}}\,. \tag{82}$$

This can be explained in the following manner.

By taking ω_{cr} as given by (82) instead of ω, in calculating the centrifugal forces as above, all these forces will increase in the ratio y_1/y_1'; the deflections graphically derived will now also increase in the same proportion and the deflection curve, as obtained graphically, will now coincide with the *initially assumed* deflection curve. This means that

* The effect of the weight of the shaft on the critical speeds will be considered later.

at a speed given by eq. (82), the centrifugal forces are sufficient to keep the rotating shaft in a deflected form. Such a speed is called a *critical speed* (see p. 164).

It was assumed in the previous discussion that the deflection curve, as obtained graphically, had deflections proportional to those of the curve initially taken. If there is a considerable difference in the shape of these two curves and a closer approximation for ω_{cr} is desired, the construction described above should be repeated by taking the deflection curve, obtained graphically, as the initial deflection curve.*

The case of a shaft on three supports and having one disc on each span (Fig. 81) will now be considered. In the solution of this problem we may proceed exactly in the same manner as before in the analytical solution and establish first the equations,

$$y_1 = a_{11}w_1 + a_{12}w_2, \qquad (a)'$$

$$y_2 = a_{21}w_1 + a_{22}w_2, \qquad (b)'$$

between the acting forces and the resultant deflections.

In order to obtain the values of the constants a_{11}, a_{12}, $\cdots$ graphically we assume first that the load w_1 is acting alone and that the middle support is removed (Fig. 85, *a*); then the deflections y_1', y_2' and y_c' can easily be obtained by using the graphical method, described before (see p. 64). Now, by using the same method, the deflection curve produced by a vertical force R' applied at C and acting in an upwards direction should be constructed and the deflections y_1'', y_c'' and y_2'' measured. Taking into consideration that the deflection at the support C should be equal to zero the reaction R of this support will now be found from the equation,

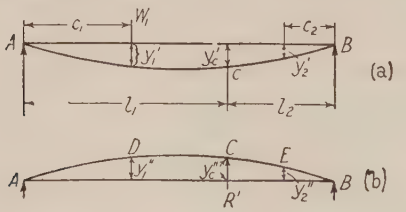


FIG. 85.

$$R = R'\frac{y_c'}{y_c''}\dagger \qquad (p)$$

* It was pointed out in considering Rayleigh's method (see paragraph (13)), that a considerable error in the shape of the assumed deflection curve produces only a small effect on the magnitude of ω_{cr} provided the conditions at the ends are satisfied.

† Absolute values of the deflections are taken in this equation.

and the actual deflections at D and E, produced by load w_1, will be

$$y_{11} = y_1' - y_1''\frac{y_c'}{y_c''},$$
$$y_{21} = y_2' - y_2''\frac{y_c'}{y_c''}. \tag{q}$$

Comparing these equations with the eqs. $(a)'$ and $(b)'$ we obtain

$$a_{11}w_1 = y_1' - y_1''\frac{y_c'}{y_c''},$$
$$a_{21}w_1 = y_2' - y_2''\frac{y_c'}{y_c''},$$

from which the constants a_{11} and a_{21} can be calculated. In the same manner, considering the load w_2, the constants a_{12} and a_{22} can be found. All constants of eqs. $(a)'$ and $(b)'$ being determined, the two critical speeds of the shaft can be calculated by using formula (81), in which $a = a_{11}$; $b = a_{12} = a_{21}$; $c = a_{22}$.

In the previous calculations, the reaction R at the middle support has been taken as the statically indeterminate quantity. In case there are many supports, it is simpler to take as statically indeterminate quantities the bending moments at the intermediate supports. To illustrate this method of calculation, let us consider a motor generator set consisting of an induction motor and a D.C. generator supported on three bearings.* The dimensions of the shaft of variable cross section are given in figure 86(a). We assume that the masses of the induction motor armature, D.C. armature and also D.C. commutator are concentrated at their centers of gravity (Fig. 86, b). In order to take into account the mass of the shaft, one-half of the mass of the left span of the shaft has been added to the mass of the induction motor and one-half of the mass of the right span of the shaft has been equally distributed between the D.C. armature and D.C. commutator. In this manner the problem is reduced to one of three degrees of freedom and the deflections y_1, y_2, y_3 of the masses W_1, W_2, and W_3 during vibration will be taken as coordinates. The statical deflections under the action of loads W_1, W_2, W_3 can be represented by eqs. (n) and the constants a_{11}, a_{12}, $\cdots$ of these eqs. will now be determined by taking the bending moment at the intermediate support as the statically indeterminate quantity. In order to obtain a_{11},

* These numerical data represent an actual case calculated by J. P. DenHartog, Research Engineer, Westinghouse Electric and Manufacturing Company, East Pittsburgh, Pennsylvania.

let us assume that the shaft is cut into two parts at the intermediate support and that the right span is loaded by a 1 lb. load at the cross section where W_1 is applied (Fig. 86, *c*). By using the graphical method,

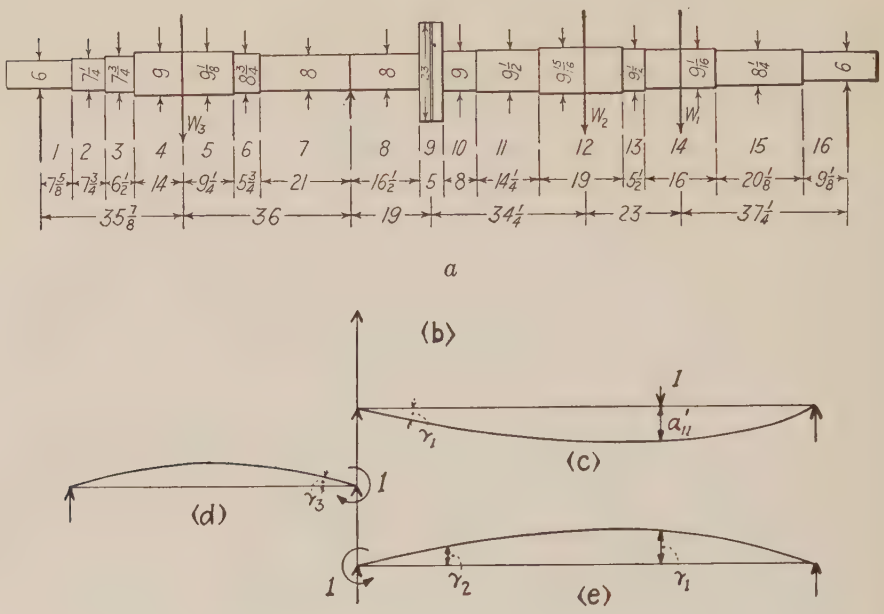


FIG. 86.

explained in paragraph 15, we obtain the deflection under the load $a_{11}' = 2.45 \times 10^{-6}$ inch and the slope at the left support $\gamma_1 = 5.95 \times 10^{-8}$ radian. By applying now a bending moment of 1 inch pound at the intermediate support and using the same graphical method, we obtain the slopes $\gamma_2 = 4.23 \times 10^{-9}$ (Fig. 86, *e*) and $\gamma_3 = 3.5 \times 10^{-9}$ (Fig. 86, *d*). From the *reciprocity theorem* it follows that the deflection at the point W_1 for this case is numerically equal to the slope γ_1, in the case shown in (Fig. 86, *c*). Combining these results it can now be concluded that the bending moment at the intermediate support produced by a load of 1 lb. at the point W_1 is

$$M = \frac{\gamma_1}{\gamma_2 + \gamma_3} \text{ lbs.} \times \text{inch}$$

and that the deflection under this load is

$$a_{11} = a_{11}' - \frac{\gamma_1^2}{\gamma_2 + \gamma_3} = 19.9 \times 10^{-7}.$$

Proceeding in the same manner with the other constants of eqs. (n) the following numerical values have been obtained:

$$a_{22} = 19.6 \times 10^{-7}; \qquad a_{33} = 7.6 \times 10^{-7}; \qquad a_{12} = a_{21} = 18.1 \times 10^{-7};$$

$$a_{13} = a_{31} = -3.5 \times 10^{-7}; \qquad a_{23} = a_{32} = -4.6 \times 10^{-7}.$$

Now substituting in eqs. (n) the centrifugal forces $w_1\omega^2y_1/g$, $w_2\omega^2y_2/g$ and $w_3\omega^2y_3/g$ instead of the loads w_1, w_2, w_3, the following eqs. will be found.

$$\left(1 - a_{11}\frac{w_1\omega^2}{g}\right)y_1 - a_{12}\frac{w_2\omega^2}{g}y_2 - a_{13}\frac{w_3\omega^2}{g}y_3 = 0,$$

$$-a_{21}\frac{w_1\omega^2}{g}y_1 + \left(1 - a_{22}\frac{w_2\omega^2}{g}\right)y_2 - a_{23}\frac{w_3\omega^2}{g}y_3 = 0,$$

$$-a_{31}\frac{w_1\omega^2}{g}y_1 - a_{32}\frac{w_2\omega^2}{g}y_2 + \left(1 - a_{33}\frac{w_3\omega^2}{g}\right)y_3 = 0.$$

If the determinant of this system of equations be equated to zero, and the quantities calculated above be used for the constants a_{11}, a_{12}, $\cdots$ the following frequency equation for calculating the critical speeds is arrived at

$$(\omega^2 10^{-7})^3 - 3.76(\omega^2 10^{-7})^2 + 1.93(\omega^2 10^{-7}) - .175 = 0,$$

from which the three critical speeds in R.P.M. are:

$$n_1 = \frac{\omega_1 60}{2\pi} = 1070; \qquad n_2 = \frac{\omega_2 60}{2\pi} = 2240; \qquad n_3 = \frac{\omega_3 60}{2\pi} = 5620.$$

In addition to the above method, the direct method of graphical solution previously described for a shaft with one span, can also be applied to the present case of two spans. In this case an initial deflection curve satisfying the conditions at the supports (Fig. 82, a, b) should be taken and a certain angular velocity ω assumed. The centrifugal forces acting on the shaft will then be

$$\frac{w_1}{g}\omega^2 y_1 \qquad \text{and} \qquad \frac{w_2}{g}\omega^2 y_2.$$

By using the graphical method the deflection curve produced by these

two forces can be constructed and if the initial curve was chosen correctly the constructed deflection curve will be geometrically similar to the initial curve and the critical speed will be obtained from an equation analogous to eq. 82. If there is a considerable difference in the shape of these two curves the construction should be repeated by considering the obtained deflection curve as the initial curve.*

This method can be applied also to the case of many discs and to cases where the mass of the shaft should be taken into consideration. We begin again by taking an initial deflection curve (Fig. 87) and by

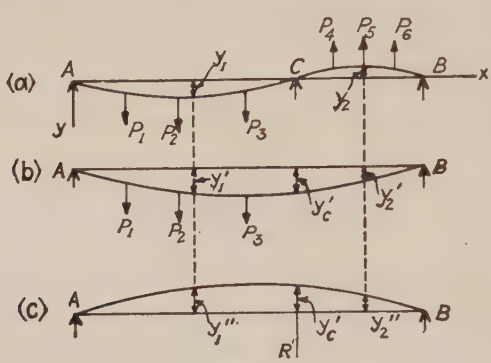


FIG. 87.

assuming a certain angular velocity ω. Then the centrifugal forces $P_1, P_2, \cdots$ acting on the discs and on portions of the shaft can easily be calculated, and the corresponding deflection curve can be constructed as follows: Consider first the forces acting on the left span of the shaft and, removing the middle support C, construct the deflection curve shown in Fig. 87, *b*. In the same manner the deflection curve produced by the vertical load R' applied at C and acting in an upwards direction can be obtained (Fig. 87, *c*) and reaction R at the middle support produced by the loading of the left span of the shaft can be calculated by using eq. (p) above. The deflection produced at any point by the loading of the left side of the shaft can then be found by using equations, similar to (q).

Taking, for instance, the cross sections in which the initial curve has the largest deflections y_1 and y_2 (Fig. 87, *a*) the deflections produced at these cross sections by the loading acting on the left side of the shaft will be

* It can be shown that this process is convergent when calculating the slowest critical speed and by repeating the construction described above we approach the true critical speed. See the book by A. Stodola, mentioned above.

$$y_{1a} = y_1' - y_1'' \frac{y_c'}{y_c''},$$

$$y_{2a} = y_2' - y_2'' \frac{y_c'}{y_c''}.$$

In the same manner the deflections y_{1b} and y_{2b} produced in these cross sections by the loading of the right side of the shaft can be obtained and the complete deflections $y_{1a} + y_{1b}$ and $y_{2a} + y_{2b}$ can be calculated.* If the initial deflection curve was chosen correctly, the following equation should be fulfilled,

$$\frac{y_{1a} + y_{1b}}{y_{2a} + y_{2b}} = \frac{y_1}{y_2}, \tag{r}$$

and the critical speed will be calculated from the equation

$$\omega_{cr} = \omega \sqrt{\frac{y_1}{y_{1a} + y_{1b}}}. \tag{83}$$

If there is a considerable deviation from condition (r) the calculation of a second approximation becomes necessary for which purpose the following procedure can be adopted.† It is easy to see that the deflections y_{1a} and y_{2a}, found above, should be proportional to ω^2 and to the initial deflection y_1, so that we can write the equations

$$y_{1a} = a_1 y_1 \omega^2,$$
$$y_{2a} = a_2 y_1 \omega^2,$$

from which the constants a_1 and a_2 can be calculated. In the same manner from the equations

$$y_{1b} = b_1 y_2 \omega^2,$$
$$y_{2b} = b_2 y_2 \omega^2,$$

the constants b_1 and b_2 can be found.

Now, if the initial deflection curve had been chosen correctly and if $\omega = \omega_{cr}$, the following equations should be satisfied

$$y_1 = y_{1a} + y_{1b} = a_1 y_1 \omega^2 + b_1 y_2 \omega^2,$$
$$y_2 = y_{2a} + y_{2b} = a_2 y_1 \omega^2 + b_2 y_2 \omega^2,$$

* Deflections in a downwards direction are taken as positive.

† This method was developed by Mr. Borowicz in his thesis "Beitrage zur Berechnung krit. Geschwindigkeiten zwei und mehrfach gelagerter Wellen.," München, 1915.

which can be written as follows:

$$\begin{aligned}(1 - a_1\omega^2)y_1 - b_1\omega^2 y_2 &= 0,\\ -a_2 y_1\omega^2 + (1 - b_2\omega^2)y_2 &= 0.\end{aligned} \tag{s}$$

The equation for calculating the critical speed will now be obtained by equating to zero the determinant of these equations, and we obtain,

$$(a_1b_2 - a_2b_1)\omega^4 - (a_1 + b_2)\omega^2 + 1 = 0.$$

The root of this equation which makes the ratio y_1/y_2 of equations (s) negative, corresponds to the assumed shape of the curve (Fig. 87, a) and gives the lowest critical speed. For obtaining a closer approximation the ratio y_1/y_2, as obtained from eqs. (s), should be used in tracing the new shape of the initial curve and with this new curve the graphical solution should be repeated. In actual cases this further approximation is usually unnecessary.

33. Gyroscopic Effects on the Critical Speeds of Rotating Shafts.—*General.*—In our previous discussion on the critical speeds of rotating shafts only the centrifugal forces of the rotating masses were taken into consideration. Under certain conditions not only these forces, but also the moments of the inertia forces due to angular movements of the axes of the rotating masses are of importance and should be taken into account in calculating the critical speeds. In the following the simplest case of a single circular disc on a shaft will be considered (Fig. 88).

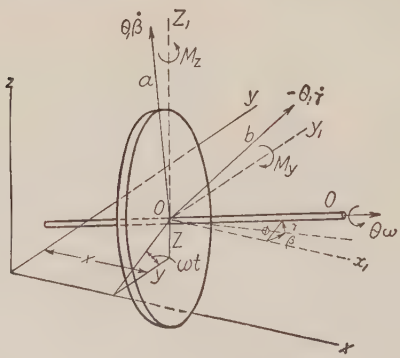


Fig. 88.

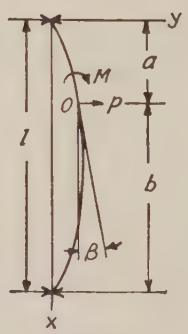


Fig. 89.

Assuming that the deflections y and z of the shaft during vibration are very small and that the center of gravity O of the disc coincides with the axis of the shaft, the position of the disc will be completely determined

by the coordinates y and z of its center and by the angles β and γ which the axis O–O perpendicular to the plane of the disc and tangent to the deflection curve of the shaft makes with the fixed planes xz and xy, perpendicular to each other and drawn through the x axis joining the centers of the bearings. Letting W equal the weight of the disc and taking into consideration the elastic reactions of the shaft * only, the equations of motion of the center of gravity of the disc will be

$$\frac{W}{g}\ddot{y} = Y, \qquad \frac{W}{g}\ddot{z} = Z, \tag{a}$$

in which Y and Z are the components of the reaction of the shaft in the y and z directions. These are linear functions of the coordinates y, z and of the angles β, γ which can be easily determined from a consideration of the bending of the shaft.

Take, for instance, the bending of a shaft with simply supported ends, in the xy plane (Fig. 89) under the action of a force P and of a couple M. Considering in the usual way the deflection curve of the shaft we obtain the deflection at O equal to †

$$y = \frac{Pa^2b^2}{3lEI} + \frac{Mab(a-b)}{3lEI}, \tag{b}$$

and the slope at the same point equal to

$$\beta = \frac{Pab(b-a)}{3lEI} - \frac{M(a^2 - ab + b^2)}{3lEI}. \tag{c}$$

From eqs. (b) and (c) we obtain

$$P = 3lEI\left(\frac{a^2 - ab + b^2}{a^3b^3}y + \frac{a-b}{a^2b^2}\beta\right), \tag{d}$$

$$M = 3lEI\left(\frac{b-a}{a^2b^2}y - \frac{1}{ab}\beta\right). \tag{e}$$

By using eq. (d) the equations (a) of motion of the center of gravity of the disc become

$$\frac{W}{g}\ddot{y} + my + n\beta = 0; \qquad \frac{W}{g}\ddot{z} + mz + n\gamma = 0, \tag{84}$$

* The conditions assumed here correspond to the case of a vertical shaft when the weight of the disc does not affect the deflections of the shaft. The effect of this weight will be considered later (see p. 188).

† See "Applied Elasticity," p. 89.

in which

$$m = 3lEI\frac{a^2 - ab + b^2}{a^3b^3}; \qquad n = 3lEI\frac{a - b}{a^2b^2}. \tag{h}$$

In considering the relative motion of the disc about its center of gravity it will be assumed that the moment of the external forces acting on the disc with respect to the O–O axis is always equal to zero, then the angular velocity ω with respect to this axis remains constant. The moments M_y and M_z, taken about the y_1 and z_1 axes parallel to the y and z axes (see Fig. 88), and representing the action of the elastic forces of the shaft on the disc can be written in the following form,

$$\begin{aligned} M_y &= -m'z + n'\gamma, \\ M_z &= m'y - n'\beta, \end{aligned} \tag{g}$$

in which m' and n' are constants which can be obtained from the deflection curve of the shaft.* The positive directions for the angles β and γ and for the moments M_y and M_z are indicated in the figure.

In the case considered above (see eq. e), we have

$$m' = 3lEI\frac{b - a}{a^2b^2}; \qquad n' = \frac{3lEI}{ab}. \tag{k}$$

The equations of relative motion of the disc with respect to its center of gravity will now be obtained by using the principle of *angular momentum* which states that the *rate of increase* of the total moment of momentum of any moving system about any *fixed* axis is equal to the total moment of the external forces about this axis. In calculating the *rate of change* of the angular momentum about a fixed axis drawn through the instantaneous position of center of gravity O we may take into consideration only the relative motion.†

In calculating the components of the angular momentum the principal axis of inertia of the disc will be taken. The axis of rotation OO is one of these axes. The two other axes will be two perpendicular diameters of the disc. One of these diameters Oa we take in the plane OOz_1 (see Fig. 88). It will make a small angle γ with the axis Oz_1. Another diameter Ob will make the angle β with the axis Oy_1.

Let θ = moment of inertia of the disc about the O–O axis,

$\theta_1 = \theta/2$ = moment of inertia of the disc about a diameter.

* It is assumed that the flexibility of the shaft including the flexibility of its supports is the same in both directions.

† See, for instance, H. Lamb, "Higher Mechanics," 1920, p. 94.

Then the component of angular momentum about the OO axis will be $\theta\omega$, and the components about the diameters Oa and Ob will be $\theta_1\dot{\beta}$ and $-\theta_1\dot{\gamma}$, respectively.* Positive directions of these components of the angular momentum are shown in Fig. 88. Projecting these components on the fixed axes Oy_1 and Oz_1 through the instantaneous position of the center of gravity O we obtain $\theta\omega\beta - \theta_1\dot{\gamma}$ and $\theta\omega\gamma + \theta_1\dot{\beta}$, respectively. Then from the principle of angular momentum we have

$$\frac{d}{dt}(\theta\omega\beta - \theta_1\dot{\gamma}) = M_y \qquad \text{and} \qquad \frac{d}{dt}(\theta\omega\gamma + \theta_1\dot{\beta}) = M_z,$$

or by using eqs. (g)

$$\begin{aligned} \theta\omega\dot{\beta} - \theta_1\ddot{\gamma} &= -m'z + n'\gamma, \\ \theta\omega\dot{\gamma} + \theta_1\dot{\beta} &= m'y - n'\beta. \end{aligned} \tag{85}$$

Four eqs. (84) and (85) describing the motion of the disc, will be satisfied by substituting

$$y = A \sin pt; \quad z = B \cos pt; \quad \beta = C \sin pt; \quad \gamma = D \cos pt. \tag{m}$$

In this manner four linear homogeneous equations in A, B, C, D will be obtained. Putting the determinant of these equations equal to zero, the equation for calculating the frequencies p of the natural vibrations will be determined.† Several particular cases will now be considered.

As a first example consider the case in which the principal axis OO perpendicular to the plane of the disc remains always in a plane containing the x axis and rotating with the constant angular velocity ω, with which the disc rotates. Putting r equal to the deflection of the shaft and φ equal to the angle between OO and x axes (see Fig. 88) we obtain for this particular case,

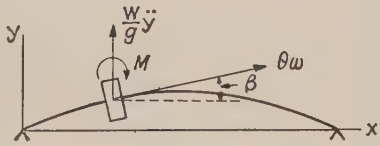


Fig. 90.

$$y = r \cos \omega t; \quad z = r \sin \omega t; \quad \beta = \varphi \cos \omega t; \quad \gamma = \varphi \sin \omega t. \tag{n}$$

Considering r and φ as constants and considering the instantaneous position when the plane of the deflected line of the shaft coincides with

* It is assumed, as before, that β and γ are small. Then $\dot{\beta}$ and $-\dot{\gamma}$ will be approximate values of the angular velocities about the diameters Oa and Ob.

† See the paper by A. Stodola in Zeitschrift f. d. gesamte Turbinenwesen, 1918, p. 253, and 1920, p. 1.

the xy plane (Fig. 90) we obtain from eqs. (n)

$$\begin{aligned} \beta &= \varphi, & \dot{\beta} &= 0, & \ddot{\beta} &= -\varphi\omega^2, \\ \gamma &= 0, & \dot{\gamma} &= \varphi\omega, & \ddot{\gamma} &= 0, \\ y &= r, & \dot{y} &= 0, & \ddot{y} &= -r\omega^2, \\ z &= 0, & \dot{z} &= r\omega, & \ddot{z} &= 0. \end{aligned}$$

Substituting in eqs. (84) and (85) we obtain,

$$\begin{aligned} \frac{W}{g}\ddot{y} + my + n\beta &= 0, \\ (\theta - \theta_1)\beta\omega^2 &= m'y - n'\beta. \end{aligned} \tag{o}$$

It is seen that the shaft is bent not only by centrifugal force but also by the moment $M = (\theta - \theta_1)\beta\omega^2$ which represents the *gyroscopic effect* of the rotating disc in this case and makes the shaft stiffer. Substituting

$$y = r\cos\omega t, \qquad \beta = \varphi\cos\omega t,$$

in eqs. (o) we obtain,

$$\begin{aligned} \left(m - \omega^2\frac{W}{g}\right)r + n\varphi &= 0, \\ -m'r + \{n' + (\theta - \theta_1)\omega^2\}\varphi &= 0. \end{aligned} \tag{p}$$

The deflection of the shaft, assumed above, becomes possible if eqs. (p) may have for r and φ roots other than zero, i.e., when the angular velocity ω is such that the determinant of these equations becomes equal to zero. In this manner the following equation for calculating the critical speeds will be found:

$$\left(m - \frac{\omega^2 W}{g}\right)\{n' + (\theta - \theta_1)\omega^2\} + nm' = 0, \tag{r}$$

or letting

$$\frac{mg}{W} = p^2; \qquad \frac{n'}{\theta - \theta_1} = q^2,$$

and noting that, from (h) and (k),

$$nm' = -cmn', \quad \text{where} \quad c = \frac{(a-b)^2}{a^2 - ab + b^2},$$

we obtain

$$(p^2 - \omega^2)(q^2 + \omega^2) - cp^2q^2 = 0$$

or

$$\omega^4 - (p^2 - q^2)\omega^2 - p^2q^2(1 - c) = 0. \tag{s}$$

It is easy to see that (for $c < 1$) eq. (s) has only one positive root for ω^2, namely,

$$\omega^2 = \tfrac{1}{2}(p^2 - q^2) + \sqrt{\tfrac{1}{4}(p^2 - q^2)^2 + (1 - c)p^2q^2}. \qquad (t)$$

When the gyroscopic effect can be neglected, $\theta - \theta_1 = 0$ should be substituted in (r) and we obtain,

$$\frac{\omega^2 W}{g} = \frac{mn' + nm'}{n'} = \frac{3lEI}{a^2b^2},$$

from which

$$\omega_{cr} = \sqrt{\frac{g3lEI}{a^2b^2W}} \qquad \text{or} \qquad \omega_{cr} = \sqrt{\frac{g}{\delta}},$$

where

$$\delta = \frac{a^2b^2W}{3lEI}$$

represents the statical deflection of the shaft under the load W. This result coincides completely with that found before (see paragraph 15) considering the disc on the shaft as a system with one degree of freedom.

In the above discussion it was assumed that the angular velocity of the plane of the deflected shaft is the same as that of the rotating disc. It is possible also that these two velocities are different. Assuming, for instance, that the angular velocity of the plane of the deflected shaft is λ and substituting,

$$y = r\cos\lambda t; \qquad z = r\sin\lambda t; \qquad \beta = \varphi\cos\lambda t; \qquad \gamma = \varphi\sin\lambda t$$

in equations (f) and (l) we obtain,

$$\frac{W}{g}\ddot{y} + my + n\beta = 0, \qquad (o)^1$$

$$(\theta\omega\lambda - \theta_1\lambda^2)\beta = m'y - n'\beta,$$

instead of eqs. (o).

By putting $\lambda = \omega$ the previous result will be obtained. If $\lambda = -\omega$ we obtain from the second of eqs. $(o)^1$

$$-(\theta + \theta_1)\omega^2\beta = m'y - n'\beta. \qquad (u)$$

This shows that when the plane of the bent shaft rotates with the velocity ω in the direction opposite to that of the rotation of the disc, the gyroscopic effect will be represented by the moment

$$M = -(\theta + \theta_1)\omega^2\beta.$$

The minus sign indicates that under such conditions the gyroscopic

moment is acting in the direction of increasing the deflection of the shaft and hence lowers the critical speed of the shaft. If the shaft with the disc is brought up to the speed ω from the condition of rest, the condition $\lambda = \omega$ usually takes place. But if there are disturbing forces of the same frequency as the critical speed for the condition $\lambda = -\omega$, then rotation of the bent shaft in a direction opposite to that of the rotating disc may take place.* If there are several discs on the shaft and the gyroscopic effect must be taken into account, the graphical method described in paragraph 32 also can be used. It is only necessary to take, in addition to the centrifugal forces, the moments due to gyroscopic effect. In Fig. 91 the case of a single disc, for the condition $\lambda = \omega$, is represented. In order to arrive at the bending moment diagram for this case, the bending moments due to centrifugal force (Fig. 91, *b*)

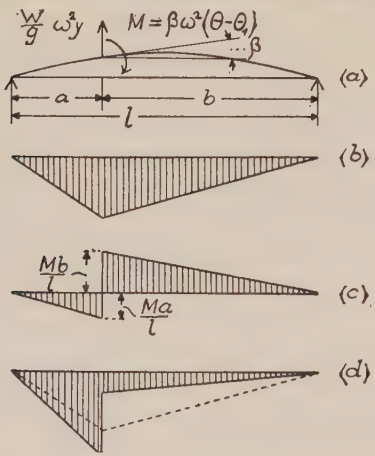


Fig. 91.

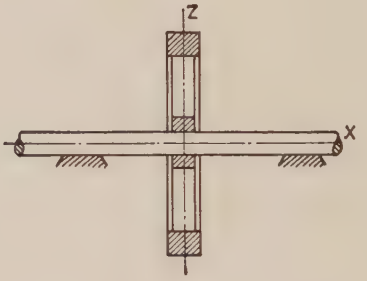


Fig. 92.

should be superimposed on the bending moments due to the couple M (Fig. 91, *c*). The resultant bending moment diagram is shown in Fig. 91, *d*. With this diagram we should proceed further exactly in the same manner as was previously explained (see p. 64).

Gyroscopic Effect in Vibration of Flywheels.—As another example consider the critical speeds of a flywheel, the spokes of which are flexible in the direction of the axis of the shaft (see Fig. 92), the shaft being absolutely rigid. Any vibration of the flywheel parallel to the axis of the shaft will not be affected by angular motion of the flywheel and the corresponding period can be calculated from eq. 5 in which δ denotes

* See A. Stodola, Dampf- und Gas Turbinen (1924), p. 367.

now the deflection of the rim of the flywheel under the action of its weight, should the shaft be put in a vertical position.

In considering the vibrations connected with angular movements of the plane of the flywheel rim the above theory will be used. The shaft being rigid, the deflections y and z in equations (85) should be taken equal to zero and we obtain

$$\begin{aligned} \theta\omega\dot{\beta} - \theta_1\ddot{\gamma} &= n'\gamma, \\ \theta\omega\dot{\gamma} + \theta_1\ddot{\beta} &= -n'\beta, \end{aligned} \tag{v}$$

or

$$\begin{aligned} \theta_1\ddot{\beta} &= -\theta\omega\dot{\gamma} - n'\beta, \\ \theta_1\ddot{\gamma} &= \theta\omega\dot{\beta} - n'\gamma, \end{aligned} \tag{86}$$

in which the spring constant n' depends on the flexibility of the spokes. This constant n' can be determined from the condition that $n'\beta$ represents the couple about the z axis necessary to produce rotation of the plane of the rim of the flywheel about this axis through an angle β.

There is an interesting analogy between the equations (86) and the equations of motion of a particle in a plane. Let O denote a fixed point, the center of gravity of the rim (Fig. 93) and OO an instan-

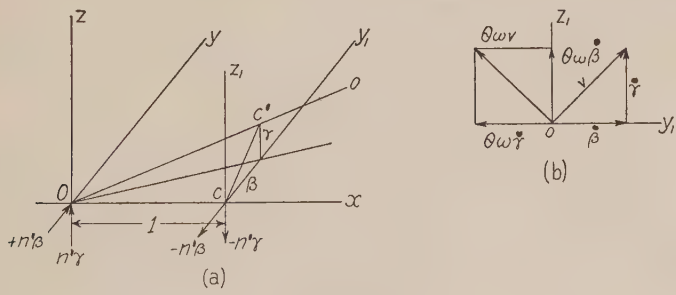


FIG. 93.

taneous position of the principal axis perpendicular to the plane of the rim. Imagine a sphere having its center at the fixed point O and a radius equal to unity. Let C be the point in which the undisturbed position of the axis of the flywheel meets the unit sphere and C' the point in which the instantaneous axis OO meets the same sphere. Considering only the cases when the angles β and γ are small, it can be assumed that point C', the "pole" of the gyroscope, moves in the plane y_1z_1 tangent to the unit sphere at point C. The external forces representing the action of the elastic forces of the spokes on the rim are given by the couples $-n'\beta$

and $-n'\gamma$ in Fig. 93. Imagine now that the pole C' is a particle having mass θ_1 and that on this particle in addition to the forces $-n'\beta$ and $-n'\gamma$, mentioned above, an imaginary "deviating force" $\theta\omega v$ proportional to the velocity v of the pole, is acting in a direction always towards the left of the path, as viewed from outside the sphere (see Fig. 93, *b*). Then the equations of motion of the pole will be exactly the same as the equations (v) above. This analogy is very useful in the solution of problems of motion of gyroscopes.

In considering vibrations of the flywheel the solutions of eqs. (v) may be taken in the form,

$$\beta = A \cos pt; \qquad \gamma = B \sin pt.$$

Substituting in (v) we obtain

$$A(n' - \theta_1 p^2) + B\theta\omega p = 0,$$
$$A\theta\omega p + B(n' - \theta_1 p^2) = 0,$$

from which the frequency equation

$$(n' - \theta_1 p^2)^2 - \theta^2\omega^2 p^2 = 0$$

is obtained.

The solution of this equation will be

$$p^2 = \frac{n'}{\theta_1}\left\{1 + \frac{\theta^2\omega^2}{2\theta_1 n'} \pm \sqrt{\left(1 + \frac{\theta^2\omega^2}{2\theta_1 n'}\right)^2 - 1}\right\}. \tag{87}$$

Fig. 94.

We obtain two different frequencies of vibration and the path of the pole C, representing the vibration of the flywheel, consists of two superimposed circular vibrations. An example of the curve described by the pole is shown in figure 94.

In the particular case when $\omega = 0$ we obtain

$$p = \sqrt{\frac{n'}{\theta_1}}.$$

Take now, as an example,*

$$\frac{\theta^2\omega^2}{2\theta_1 n'} = 2.$$

Then, from eq. (87)

$$p = \sqrt{\frac{n'}{\theta_1}}\sqrt{3 \pm \sqrt{8}}$$

* These proportions correspond to an actual case of a flywheel with flexible spokes.

and we obtain

$$p_1 = 2.415\sqrt{\frac{n'}{\theta_1}}; \qquad p_2 = .415\sqrt{\frac{n'}{\theta_1}}.$$

The path of the pole C will be the superposition of two harmonic vibrations of the periods, $2\pi/p_1$ and $2\pi/p_2$. If external periodic forces are acting on the flywheel forced vibrations will be produced. Assuming, for instance, that a simple harmonic force of the same period as the revolutions of the machine is acting, then instead of eqs. (86) the following equations will be obtained.

$$\begin{aligned} \theta_1\ddot{\beta} &= -\theta\omega\dot{\gamma} - n'\beta + P\cos\omega t, \\ \theta_1\ddot{\gamma} &= \theta\omega\dot{\beta} - n'\gamma + P\sin\omega t. \end{aligned} \tag{w}$$

The forced vibration of the flywheel will be represented by the particular solution of these equations. Assuming

$$\beta = A\cos\omega t; \qquad \gamma = B\sin\omega t,$$

and substituting in eqs. (w) we obtain

$$\begin{aligned} A(n' - \theta_1\omega^2) + B\theta\omega^2 &= P, \\ A\theta\omega^2 + B(n' - \theta_1\omega^2) &= P, \end{aligned}$$

from which

$$A = \frac{P\{n' - \omega^2(\theta + \theta_1)\}}{(n' - \theta_1\omega^2)^2 - \theta^2\omega^4}.$$

Conditions of resonance occur when

$$(n' - \theta_1\omega^2)^2 - \theta^2\omega^4 = 0,$$

from which

$$\omega^2 = \frac{n'}{\theta_1\left(\frac{\theta^2}{\theta_1^2} - 1\right)}\left(-1 \pm \frac{\theta}{\theta_1}\right).$$

The only real root giving the critical speed for this case is,

$$\omega_{cr} = \sqrt{\frac{n'}{\theta_1\left(\frac{\theta}{\theta_1} + 1\right)}}. \tag{88}$$

Vibration of a Rigid Rotor with Flexible Bearings.—Equations (84) and (85) can be used also in the study of vibration of a rigid rotor, having bearings in flexible pedestals (Fig. 95). Let y_1z_1 and y_2z_2 be small displacements of the bearings during vibration. Taking these displace-

ments as coordinates of the oscillating rotor, the displacements of the center of gravity and the angular displacements of the axis of the rotor will be (see Fig. 95).

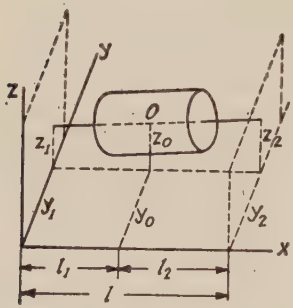


FIG. 95.

$$y_0 = y_1\frac{l_2}{l} + y_2\frac{l_1}{l},$$

$$z_0 = z_1\frac{l_2}{l} + z_2\frac{l_1}{l},$$

$$\beta = \frac{y_2 - y_1}{l},$$

$$\gamma = \frac{z_2 - z_1}{l}.$$

Let c_1, c_2, d_1 and d_2 be constants depending on the flexibility of the pedestals in the horizontal and vertical directions, such that $-c_1y_1$, $-c_2y_2$ are horizontal and $-d_1z_1$, $-d_2z_2$ are the vertical reactions of the bearings due to the small displacements y_1, y_2, z_1 and z_2 in the y and z directions. Then the equations of motion of the center of gravity (84) become

$$\begin{aligned} \frac{W}{gl}(l_2\ddot{y}_1 + l_1\ddot{y}_2) + c_1y_1 + c_2y_2 &= 0, \\ \frac{W}{gl}(l_2\ddot{z}_1 + l_1\ddot{z}_2) + d_1z_1 + d_2z_2 &= 0. \end{aligned} \qquad (a)^1$$

The equations (85), representing the rotations of the rotor about the y and z axis will be in this case

$$\begin{aligned} \theta\omega\frac{\dot{y}_2 - \dot{y}_1}{l} - \theta_1\frac{\ddot{z}_2 - \ddot{z}_1}{l} &= z_2d_2l_2 - z_1d_1l_1, \\ \theta\omega\frac{\dot{z}_2 - \dot{z}_1}{l} + \theta_1\frac{\ddot{y}_2 - \ddot{y}_1}{l} &= -y_2c_2l_2 + y_1c_1l_1. \end{aligned} \qquad (b)^1$$

The four equations $(a)^1$ and $(b)^1$ completely describe the free vibrations of a rigid rotor on flexible pedestals. Substituting in these equations

$$y_1 = A \sin pt; \quad y_2 = B \sin pt; \quad z_1 = C \cos pt; \quad z_2 = D \cos pt,$$

four homogeneous linear equations in A, B, C, and D will be obtained. Equating the determinant of these equations to zero, we get the frequency equation from which the frequencies of the four natural modes of vibration of the rotor can be calculated.

Consider now a forced vibration of the rotor produced by some eccen-

trically attached mass. The effect of such an unbalance will be equivalent to the action of a disturbing force with the components

$$Y = A \cos \omega t; \qquad Z = B \sin \omega t,$$

applied to the center of gravity and to a couple with the components,

$$M_y = C \sin \omega t; \qquad M_z = D \cos \omega t.$$

Instead of the equations $(a)^1$ and $(b)^1$ we obtain

$$\begin{aligned}
&\frac{W}{gl}(l_2\ddot{y}_1 + l_1\ddot{y}_2) + c_1y_1 + c_2y_2 = A \cos \omega t,\\
&\frac{W}{gl}(l_2\ddot{z}_1 + l_1\ddot{z}_2) + d_1z_1 + d_2z_2 = B \sin \omega t,\\
&\theta\omega\frac{\dot{y}_2 - \dot{y}_1}{l} - \theta_1\frac{\ddot{z}_2 - \ddot{z}_1}{l} = z_2d_2l_2 - z_1d_1l_1 + C \sin \omega t,\\
&\theta\omega\frac{\dot{z}_2 - \dot{z}_1}{l} + \theta_1\frac{\ddot{y}_2 - \ddot{y}_1}{l} = -y_2c_2l_2 + y_1c_1l_1 + D \cos \omega t.
\end{aligned} \qquad (c)^1$$

The particular solution of these equations representing the forced vibration of the rotor will be of the form

$$y_1 = A \cos \omega t; \quad y_2 = B \cos \omega t; \quad z_1 = C \sin \omega t; \quad z_2 = D \sin \omega t.$$

Substituting in eqs. $(c)^1$, the amplitude of the forced vibration will be found. During this vibration the axis of the rotor describes a surface given by the equations

$$\begin{aligned}
y &= (a + bx) \cos \omega t,\\
z &= (c + dx) \sin \omega t,
\end{aligned}$$

in which a, b, c and d are constants. We see that every point of the axis describes an ellipse given by the equation,

$$\frac{y^2}{(a + bx)^2} + \frac{z^2}{(c + dx)^2} = 1.$$

For two points of the axis, namely, for

$$x_1 = -\frac{a}{b} \qquad \text{and} \qquad x_2 = -\frac{c}{d},$$

the ellipses reduce to straight lines and the general shape of the surface described by the axis of the rotor will be as shown in Fig. 96. It is seen that the displacements of a point on the axis of the rotor depend not only upon the magnitude of the disturbing force (amount of unbalance) but

also upon the position of the point along the axis and on the direction in which the displacement is measured.

In the general case the unbalance can be represented by two eccentrically attached masses (see paragraph 10) and the forced vibrations

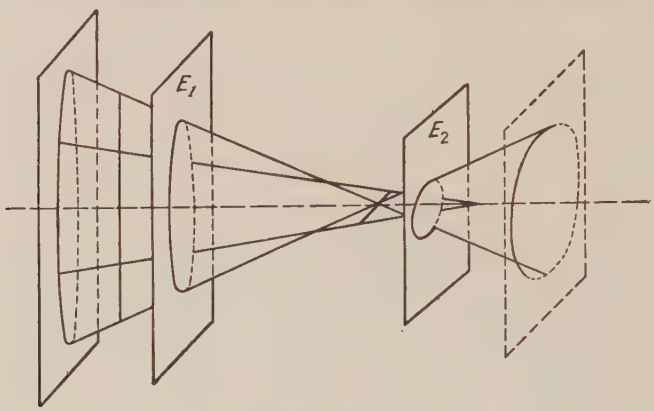


FIG. 96.

of the rotor can be obtained by superimposing two vibrations of such kind as considered above and having a certain difference in phase.* From the linearity of the equations $(c)^1$ it can also be concluded that by putting correction weights in two planes the unbalance always can be removed; it is only necessary to determine the correction weights in such a manner that the corresponding centrifugal forces will be in equilibrium with the disturbing forces due to unbalance.†

34. Effect of Weight of Shaft and Discs on the Critical Speed.—In our previous discussion the effect of the weight of the rotating discs themselves was excluded by assuming that the axis of the shaft is vertical. In the case of horizontal shafts the weights of the discs must be considered as disturbing forces which at a certain speed produce considerable vibration in the shaft. This speed is usually called "critical speed of the second order." ‡ For determining this speed a more detailed study of the motion of discs is necessary. In the following the simplest case of a

* This question is discussed in detail in the paper by V. Blaess, "Über den Massenausgleich raschumlaufender Körper," Z. f. angewandte Mathematik und Mechanik, Vol. 6 (1926), p. 429.

† The effect of flexibility of the shaft will be considered later (see paragraph 35).

‡ A. Stodola was the first to discuss this problem. The literature on the subject can be found in his book, 6th Ed., p. 929. See also the paper by T. Pöschl in Zeitschr. f. angew. Mathem. u. Mech., Vol. 3 (1923), p. 297.

single disc will be considered and it will be assumed that the disc is attached to the shaft at the cross section in which the tangent to the deflection curve of the shaft remains parallel to the center line of the bearings. In this manner the "gyroscopic effect," discussed in the previous paragraph, will be excluded and only the motion of the disc in its own plane needs to be considered. Let xy represent the horizontal plane of the disc and O the center of the vertical shaft in its undeflected position (see Fig. 97). During the vibration let S be the instantaneous position of the center of the shaft and C, the instantaneous position of the center of gravity of the disc so that $CS = e$ represents the eccentricity with which the disc is attached to the shaft. Other notations will be as follows:

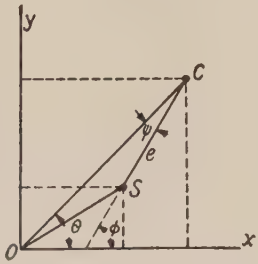


FIG. 97.

m = the mass of the disc.

mi^2 = moment of inertia of the disc about the axis through C and perpendicular to the disc.

α = spring constant of the shaft equal to the force in the xy plane necessary to produce unit deflection in this plane.

$\omega_{cr} = \sqrt{(\alpha/m)}$ = the critical speed of the first order (see paragraph 15).

x, y = coordinates of the center of gravity C of the disc during motion.

φ = the angle of rotation of the disc equal to the angle between the radius SC and x axis.

θ = the angle of rotation of the vertical plane OC.

ψ = the angle of rotation of the disc with respect to the plane OC.

Then $\varphi = \psi + \theta$. The coordinates x and y of the center of gravity C and the angle of rotation φ will be taken as coordinates determining the position of the disc in the plane xy.

The differential equations of motion of the center of gravity C can easily be written in the usual way if we note that only one force, the elastic reaction of the shaft, is acting on the disc in the xy plane. This force is proportional to the deflection OS of the shaft and its components in the x and y directions, proportional to the coordinates of the point S, will be $-\alpha(x - e\cos\varphi)$ and $-\alpha(y - e\sin\varphi)$ respectively. Then the differential equations of motion of the center C will be

$$m\ddot{x} = -\alpha(x - e\cos\varphi); \qquad m\ddot{y} = -\alpha(y - e\sin\varphi)$$

or

$$m\ddot{x} + \alpha x = \alpha e \cos \varphi,$$
$$m\ddot{y} + \alpha y = \alpha e \sin \varphi. \qquad (a)$$

The third equation will be obtained by using the principle of angular momentum. The angular momentum of the disc about the O axis consists (1) of the angular momentum $mi^2\dot{\varphi}$ of the disc rotating with the angular velocity $\dot{\varphi}$ about its center of gravity and (2) of the angular momentum $m(x\dot{y} - y\dot{x})$ of the mass m of the disc concentrated at its center of gravity. Then the principle of angular momentum gives the equation

$$\frac{d}{dt}\{mi^2\dot{\varphi} + m(x\dot{y} - y\dot{x})\} = M$$

or

$$mi^2\ddot{\varphi} + m(x\ddot{y} - y\ddot{x}) = M, \qquad (b)$$

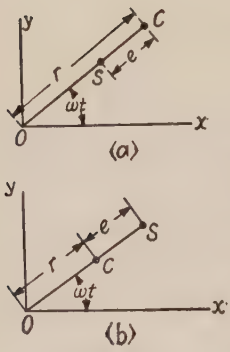


Fig. 98.

in which M is the torque moment transmitted to the disc by the shaft.

The equations (a) and (b) completely describe the motion of the disc. When $M = 0$ a particular solution of the equations (a) and (b) will be obtained by assuming that the center of gravity C of the disc remains in the plane OS of the deflection curve of the shaft and describes while rotating at constant angular velocity $\dot{\varphi} = \omega$, a circle of radius r. Then substituting in equations (a) $x = r \cos \omega t$; $y = r \sin \omega t$ and taking $\varphi = \omega t$ for the case represented in Fig. 98, a, and $\varphi = \omega t + \pi$ for the case represented in Fig. 98, b, we obtain

$$r_0 = \frac{\alpha e}{\alpha - m\omega^2} = \frac{e\omega_{cr}^2}{\omega_{cr}^2 - \omega^2} \qquad \text{for} \qquad \omega < \omega_{cr},$$
$$r_0 = -\frac{\alpha e}{\alpha - m\omega^2} = \frac{e\omega_{cr}^2}{\omega^2 - \omega_{cr}^2} \qquad \text{for} \qquad \omega > \omega_{cr}.$$

These results coincide with those obtained before from elementary considerations (see paragraph 15).

Let us now consider the case when the torque moment M is different from zero and such that *

$$M = m(x\ddot{y} - y\ddot{x}). \qquad (c)$$

* This case is discussed in detail in the dissertation "Die kritischen Zustände zweiter Art rasch umlaufender Wellen," by P. Schröder, Stuttgart, 1924. This paper contains very complete references to the new literature on the subject.

Then from eq. (*b*) we conclude that

$$\dot{\varphi} = \omega = \text{const.},$$

and by integrating we obtain

$$\varphi = \omega t + \varphi_0, \tag{d}$$

in which φ_0 is an arbitrary constant determining the initial magnitude of the angle φ.

Substituting (*d*) in eqs. (*a*) and using the notation $\omega_{cr}^2 = \alpha/m$, we obtain

$$\begin{aligned} \ddot{x} + \omega_{cr}^2 x &= \omega_{cr}^2 e \cos(\omega t + \varphi_0), \\ \ddot{y} + \omega_{cr}^2 y &= \omega_{cr}^2 e \sin(\omega t + \varphi_0). \end{aligned} \tag{e}$$

It is easy to show by substitution that

$$\begin{aligned} x &= -\frac{M_1}{e\alpha}\cos(\omega_{cr}t + \gamma_1 + \varphi_0) + \frac{e\omega_{cr}^2}{\omega_{cr}^2 - \omega^2}\cos(\omega t + \varphi_0), \\ y &= -\frac{M_1}{e\alpha}\sin(\omega_{cr}t + \gamma_1 + \varphi_0) + \frac{e\omega_{cr}^2}{\omega_{cr}^2 - \omega^2}\sin(\omega t + \varphi_0), \end{aligned} \tag{f}$$

represent a solution of the eqs. (*e*).

Substituting (*f*) in eq. (*c*) we obtain

$$M = M_1 \sin\{(\omega_{cr} - \omega)t + \gamma_1\}. \tag{g}$$

It can be concluded that under the action of the *pulsating moment* (*g*) the disc is rotating with a constant angular velocity and in the same time its center of gravity performs a combined oscillatory motion represented by the eqs. (*f*).

In the same manner it can be shown that under the action of a pulsating moment

$$M = M_2 \sin\{(\omega_{cr} + \omega)t + \gamma_2\},$$

the disc also rotates with a constant speed ω and its center performs oscillatory motions given by the equations

$$\begin{aligned} x &= \frac{M_2}{e\alpha}\cos(\omega_{cr}t + \gamma_2 - \varphi_0) + \frac{e\omega_{cr}^2}{\omega_{cr}^2 - \omega^2}\cos(\omega t + \varphi_0), \\ y &= -\frac{M_2}{e\alpha}\sin(\omega_{cr}t + \gamma_2 - \varphi_0) + \frac{e\omega_{cr}^2}{\omega_{cr}^2 - \omega^2}\sin(\omega t + \varphi_0). \end{aligned} \tag{h}$$

Combining the solutions (*f*) and (*h*) the complete solution of the eqs. (*e*), containing four arbitrary constants M_1, M_2, γ_1 and γ_2 will be

obtained. This result can now be used for explaining the vibrations produced by the weight of the disc itself.

Assume that the shaft is in a horizontal position and the y axis is upwards, then by adding the weight of the disc we will obtain Fig. 99, instead of Fig. 97. The equations (a) and (b) will be replaced in this case by the following system of equations:

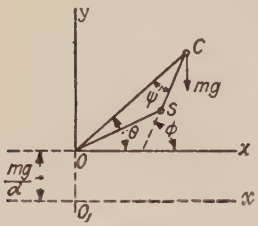


FIG. 99.

$$
\begin{aligned}
m\ddot{x} + \alpha x &= \alpha e \cos \varphi, \\
m\ddot{y} + \alpha y &= \alpha e \sin \varphi - mg, \\
mi^2\ddot{\varphi} + m(x\ddot{y} - r\ddot{x}) &= M - mgx.
\end{aligned}
\tag{k}
$$

Let us displace the origin of coordinates from O to O_1 as shown in the figure; then by letting

$$y_1 = y + \frac{mg}{\alpha},$$

eqs. (k) can be represented in the following form:

$$
\begin{aligned}
m\ddot{x} + \alpha x &= \alpha e \cos \varphi, \\
m\ddot{y}_1 + \alpha y_1 &= \alpha e \sin \varphi, \\
mi^2\ddot{\varphi} + m(x\ddot{y}_1 - y_1\ddot{x}) &= M - mge \cos \varphi.
\end{aligned}
\tag{l}
$$

This system of equations concides with the system of eqs. (a) and (b) and the effect of the disc's weight is represented by the pulsating moment $-mge \cos \varphi$. Imagine now that $M = 0$ and that the shaft is rotating with a constant angular velocity $\omega = \frac{1}{2}\omega_{cr}$. Then the effect of the weight of the disc can be represented in the following form

$$
\begin{aligned}
-mge \cos \varphi = -mge \cos (\omega t) &= mge \sin (\omega t - \pi/2) \\
&= mge \sin \{(\omega_{cr} - \omega)t - \pi/2\}. \qquad (m)
\end{aligned}
$$

This disturbing moment has exactly the same form as the pulsating moment given by eq. (g) and it can be concluded that at the speed $\omega = \frac{1}{2}\omega_{cr}$, the pulsating moment due to the weight of the disc will produce vibrations of the shaft given by the equations (f). This is the so-called *critical speed of the second order*, which in many actual cases has been observed.* It should be noted, however, that vibrations of the same frequency can be produced also by variable flexibility of the shaft (see p. 91) and it is quite possible that in some cases where a critical speed of the second order has been observed the vibrations were produced by this latter cause.

* See, O. Foeppl, V.D.I., Vol. 63 (1919), p. 867.

35. Effect of Flexibility of Shafts on the Balancing of Machines.—In our previous discussion on the balancing of machines (see paragraph 10) it was assumed that the rotor was an absolutely rigid body. In such a case complete balancing may be accomplished by putting correction weights in two arbitrarily chosen planes. The assumption neglecting the flexibility of the shaft is accurate enough at low speeds but for high speed machines and especially in the cases of machines working above the critical speed the deflection of the shaft may have a considerable effect and as a result of this, the rotor can be balanced only for one definite speed or at certain conditions cannot be balanced at all and will always give vibration troubles.

The effect of the flexibility of the shaft will now be explained on a simple example of a shaft supported at the ends and carrying two discs (see Fig. 100). The deflection of the shaft y_1 under a load W_1 will depend not only on the magnitude of this load, but also on the magnitude of the load W_2. The same conclusion holds also for the deflection y_2 under the load W_2. By using the equations of the deflection curve of a shaft on two supports, the following expressions for the deflections can be easily obtained:

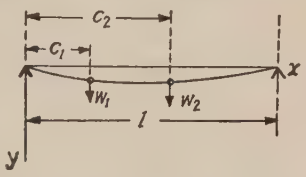


FIG. 100.

$$\begin{aligned} y_1 &= a_{11}W_1 + a_{12}W_2, \\ y_2 &= a_{21}W_1 + a_{22}W_2, \end{aligned} \qquad (a)$$

in which a_{11}, a_{12}, a_{21} and a_{22} remain constant for a given shaft and a given position of loads. These equations can be used now in calculating the deflections produced in the shaft by the centrifugal forces due to eccentricities of the discs.

Let m_1, m_2 = masses of discs I and II,
ω = angular velocity,
y_1, y_2 = deflections at the discs I and II, respectively,
c_1, c_2 = distances from the left support to the discs I and II,
Y_1, Y_2 = centrifugal forces acting on the shaft.

Assuming that only disc I has a certain eccentricity e_1 and taking the deflection in the plane of this eccentricity, the centrifugal forces acting on the shaft will be

$$Y_1 = (e_1 + y_1)m_1\omega^2; \qquad Y_2 = y_2m_2\omega^2,$$

or, by using equations similar to eqs. (a), we obtain

$$Y_1 = e_1 m_1 \omega^2 + m_1 \omega^2 (a_{11} Y_1 + a_{12} Y_2),$$
$$Y_2 = m_2 \omega^2 (a_{21} Y_1 + a_{22} Y_2),$$

from which

$$Y_1 = \frac{e_1 m_1 \omega^2 (1 - a_{22} m_2 \omega^2)}{(1 - a_{11} m_1 \omega^2)(1 - a_{22} m_2 \omega^2) - m_1 m_2 a_{12} a_{21} \omega^4},$$
$$Y_2 = \frac{e_1 a_{21} m_1 m_2 \omega^4}{(1 - a_{11} m_1 \omega^2)(1 - a_{22} m_2 \omega^2) - m_1 m_2 a_{12} a_{21} \omega^4}. \qquad (b)$$

It is seen that instead of a centrifugal force $e_1 m_1 \omega^2$, which we have in the case of a rigid shaft, two forces Y_1 and Y_2 are acting on the flexible shaft. The unbalance will be the same as in the case of a rigid shaft on which a force $R_1 = Y_1 + Y_2$ is acting at the distance from the left support equal to

$$l_1 = \frac{Y_1 c_1 + Y_2 c_2}{Y_1 + Y_2}. \qquad (c)$$

It is easy to see by using eqs. (b) that l_1 does not depend on the amount of eccentricity e_1, but only on the elastic properties of the shaft, the position and magnitude of the masses m_1 and m_2 and on the speed ω of the machine.

In the same manner as above the effect of eccentricity in disc II can be discussed and the result of eccentricities in both discs can be obtained by the principle of superposition. From this it can be concluded that at a given speed the unbalance in two discs on a flexible shaft is dynamically equivalent to unbalances in two definite planes of a rigid shaft. The position of these planes can be determined by using eq. (c) for one of the planes and an analogous equation for the second plane.

Similar conclusions can be made for a flexible shaft with any number n of discs * and it can be shown that the unbalance in these discs is equivalent to the unbalance in n definite planes of a rigid shaft. These planes remaining fixed at a given speed of the shaft, the balancing can be accomplished by putting correction weights in two planes arbitrarily chosen. At any other speed the planes of unbalance in the equivalent rigid shaft change their position and the rotor goes out of balance. This gives us an explanation why a rotor perfectly balanced in a balancing machine at a comparatively low speed may become out of balance at

* A general investigation of the effect of flexibility of the shaft on the balancing can be found in the paper by V. Blaess, mentioned before (see p. 184). From this paper the figures 101 and 102 have been taken.

service speed. Thus balancing in the field under actual conditions becomes necessary. The displacements of the planes of unbalance with variation in speed is shown below for two particular cases. In Fig. 101 a shaft carrying three discs is represented. The changes with the speed in the distances l_1, l_2, l_3 of the planes of unbalance in the equivalent rigid shaft are shown in the figure by the curves l_1, l_2, and l_3. It is seen that

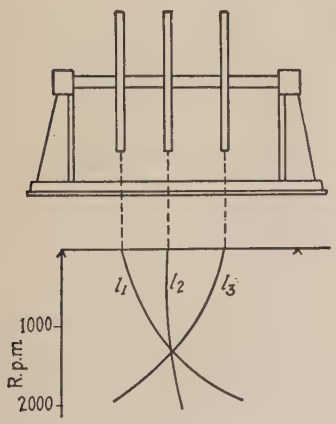


Fig. 101.

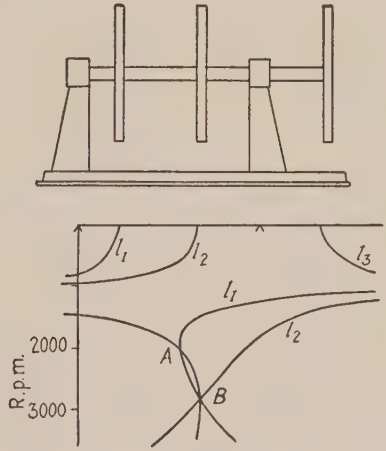


Fig. 102.

with an increase in speed these curves first approach each other, then go through a common point of intersection at the critical speed and above it diverge again. Excluding the region near the critical speed, the rotor can be balanced by at any other speed, putting correction weights in any two of the three discs. More difficult conditions are shown in Fig. 102. It is seen that at a speed equal to about 2150 r.p.m. the curves l_1 and l_3 go through the same point A. The two planes of the equivalent rigid shaft coincide and it becomes impossible to balance the machine by putting correction weights in the discs 1 and 3. Practically in a considerable region near the point A the conditions will be such that it will be difficult to obtain satisfactory balancing and heavy vibration troubles should be expected.

36. The Theory of the Dynamic Vibration Absorber.—The *vibration absorber* which will be discussed below consists of a small vibratory system tuned to the operating frequency of a larger machine and attached to it in a suitable location.* In this manner a periodic force equal in

* See paper by J. Ormondroyd and J. P. DenHartog, Trans. Amer. Soc. Mech. Eng., 1928.

frequency and magnitude to the disturbing force but acting in the opposite direction to that force may be imposed on the machine and the corresponding vibration can be eliminated. The action of the vibration absorber will now be explained on some very simple elastic systems shown in Fig. 103.

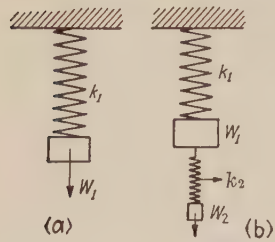


FIG. 103.

The Absorber Without Damping.—If a weight W_1 is attached to a spring whose spring constant is k_1 (Fig. 103, *a*), the period of vibration of such a system will be $\tau = 2\pi\sqrt{(W_1/k_1 g)}$. Let a disturbing force $Q_0 \cos mt$, having the period $\tau_1 = (2\pi/m)$, act on the mass W_1. Then the steady forced vibration produced by this force will be (see p. 9)

$$x = \frac{Q_0}{k_1} \frac{1}{1 - \dfrac{\tau^2}{\tau_1^2}} \cos mt. \tag{a}$$

This vibration may have a considerable amplitude provided the ratio τ/τ_1 approaches unity. In order to improve conditions, let us attach to the large load W_1 a small weight W_2 by a spring with a spring constant k_2 (Fig. 103, *b*). In this manner a vibration system with two degrees of freedom will be obtained and the forced vibration (*a*) can be eliminated by using certain proportions for k_2 and W_2. Considering the vertical displacements x_1 and x_2 of the loads W_1 and W_2 as the coordinates, the kinetic and the potential energies of the system will be

$$T = \frac{W_1}{2g}\dot{x}_1^2 + \frac{W_2}{2g}\dot{x}_2^2; \qquad V = \frac{k_1 x_1^2}{2} + \frac{k_2(x_2 - x_1)^2}{2}, \tag{b}$$

and Lagrange's eqs. (72) become

$$\begin{aligned} \frac{W_1}{g}\ddot{x}_1 + k_1 x_1 - k_2(x_2 - x_1) &= Q_0 \cos mt, \\ \frac{W_2}{g}\ddot{x}_2 + k_2(x_2 - x_1) &= 0. \end{aligned} \tag{c}$$

The steady state of forced vibration will be obtained by taking the solution of eqs. (*c*) in the form $x_1 = A \cos mt$, $x_2 = B \cos mt$. Substituting this in eqs. (*c*) we obtain

$$x_1 = Q_0 \frac{k_2 - m^2 \dfrac{W_2}{g}}{\left(k_1 + k_2 - \dfrac{W_1}{g} m^2\right)\left(k_2 - \dfrac{W_2}{g} m^2\right) - k_2^2} \cos mt,$$

$$x_2 = Q_0 \frac{k_2}{\left(k_1 + k_2 - \dfrac{W_1}{g} m^2\right)\left(k_2 - \dfrac{W_2}{g} m^2\right) - k_2^2} \cos mt. \qquad (d)$$

It is seen that vibration of the mass W_1 will be eliminated by taking

$$k_2 - m^2 \frac{W_2}{g} = 0, \qquad (e)$$

from which

$$m = \sqrt{\frac{k_2 g}{W_2}}, \qquad (f)$$

i.e., the frequency of the attached small vibrating system must be equal to the frequency of the disturbing force. Then, from eqs. (d),

$$x_1 = 0; \qquad x_2 = -\frac{Q_0}{k_2} \cos mt. \qquad (g)$$

It is seen that the motion x_2 of the load W_2 is such that the spring force acting on W_1 is $k_2 x_2 = -Q_0 \cos mt$, which is exactly equal and opposite to the impressed force. In this manner the suspension of the additional weight W_2 in the original system completely eliminates the motion of W_1, provided eq. (e) is satisfied. It should be noted that in practical applications not only the ratio between the k_2 and W_2 but also their absolute values are of importance. From the second of eqs. (g) is seen that when k_2 is taken too small, x_2 becomes large and the stresses in the spring may become excessive or the limits of possible motion x_2 may be passed. Both equations (e) and (g) must be considered in the practical design of an absorber, and the value of Q_0 will determine the smallest possible values of k_2 and W_2.

The described method of eliminating vibration may be used also in the case of torsional systems shown in figure (104). A system consisting of two masses with the moments of inertia I_1, I_2 and a shaft with a spring constant k, has a period of natural vibration equal to (see eq. 15)

$$\tau = 2\pi \sqrt{\frac{1}{k} \frac{I_1 I_2}{I_1 + I_2}}. \qquad (h)$$

If a pulsating torque $M_t \cos mt$ is acting on the mass I_1 the forced torsional

vibration of both discs I_1 and I_2 produced by this torque can be eliminated by attaching to I_1 a small vibrating system consisting of a disc with a moment of inertia i and a shaft with a spring constant k_1 (Fig. 104, *b*). It is only necessary to take for k_1 and i such proportions as to make the frequency of the attached system equal to the frequency of the pulsating torque. When this torque is in resonance with the main system, the condition which must be satisfied is

$$\frac{k_1}{i} = \frac{k(I_1 + I_2)}{I_1 I_2}. \tag{i}$$

Then the angles of rotation during vibration of discs I_1, I_2 and i will be

$$\varphi_1 = \varphi_2 = 0; \qquad \varphi_3 = -\frac{M_t}{k_1}\cos mt,$$

and the spring moment in the attached small shaft will be exactly equal and opposite to the impressed torque $M_t \cos mt$. So far, the action of

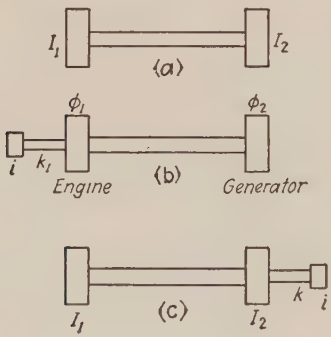


Fig. 104.

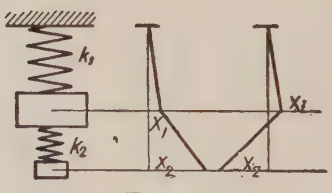


Fig. 105.

the absorber has been discussed for one frequency only. The general case of vibration of the system is given by eqs. (*c*) and it can be concluded from these equations that the system has two types of natural vibration. The frequencies of these vibrations will be determined by the two roots m_1^2 and m_2^2 of the equation, quadratic in m^2, which will be obtained by putting the denominator of the expressions (*d*) equal to zero. The ratio x_1/x_2 for these two frequencies will then be (from eqs. *d*).

$$\left(\frac{x_1}{x_2}\right)_1 = 1 - \frac{m_1^2 W_2}{g k_2}; \qquad \left(\frac{x_1}{x_2}\right)_2 = 1 - \frac{m_2^2 W_2}{g k_2}.$$

The corresponding types of vibrations for a particular case are shown schematically in Fig. 105. When the frequency of the disturbing force is

such as is given by eq. (f), the absorber eliminates vibration. But for any other frequency of the disturbing force vibration will not be eliminated and when $m = m_1$, or $m = m_2$, a condition of resonance takes place. From this it is seen that the applicability of the absorber without damping is restricted to machines with constant speed such as for instance electric synchronous or induction machines. One application of the absorber is shown in Fig. 106, which represents the outboard generator bearing pedestal of a 30,000 KW. turbo-generator. This pedestal vibrated considerably at 1800 R.P.M. in the direction of the generator axis. By bolting to the pedestal two vibration absorbers consisting of two cantilevers 20 in. long and 7/8 in. x 2 5/8 in. in cross section, weighted at the end with 25 lbs., the amplitude was reduced to about one third of its previous magnitude.

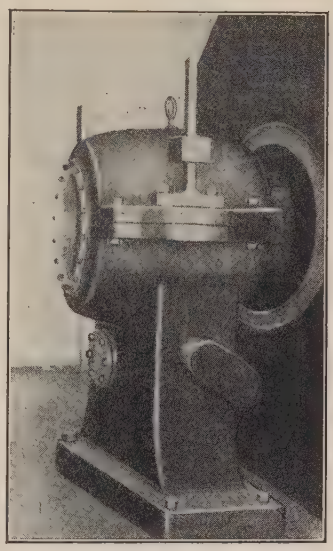


Fig. 106.

Damped Vibration Absorber.—In order to make the absorber effective over an extended range of frequencies it is necessary to introduce damping in the vibrator. From Fig. 105 is seen that the mass of the vibrator vibrates much more violently than the main mass, hence damping proportional to the velocity will be much more pronounced when located in the vibrator system, than when located in the main system. Assume, for instance, that damping is located between the masses W_1 and W_2 (Fig. 103, b) and that it is proportional to their relative velocity $\dot{x}_1 - \dot{x}_2$. Then by taking into consideration the friction force the differential equations of motion will be obtained from eqs. (c), in the following form:

$$\begin{aligned} \frac{W_1}{g}\ddot{x}_1 + k_1x_1 - k_2(x_2 - x_1) &= Q_0 \cos mt + \alpha(\dot{x}_2 - \dot{x}_1), \\ \frac{W_2}{g}\ddot{x}_2 + k_2(x_2 - x_1) &= \alpha(\dot{x}_1 - \dot{x}_2), \end{aligned} \tag{k}$$

in which the factor α depends upon the damping condition. By making this factor very small we approach the condition of a vibration absorber without damping. The action in that case has been explained above. In the case of an infinitely large α, there will exist no relative motion

between the two masses, the vibration absorber will not work, and the main mass may be put in strong vibration under the condition of resonance. For each particular case there exists a certain value of α for which the amplitude of forced vibration becomes a minimum. This optimum damping can be calculated from the solution of eqs. (k), which in the particular case when $k_1g/W_1 = k_2g/W_2$, gives the following expression for the amplitude of the forced vibration: *

$$\frac{x_1}{x_{st}} = p\sqrt{\frac{\left(\frac{\alpha^2}{m_0^2W_1^2}\right)\beta g^2p^2 + (1-\beta)^2}{\left(\frac{\alpha^2}{m_0^2W_1^2}\right)\frac{p^2g^2}{\beta}[p(1-\beta)+1]^2 + \frac{1}{\beta^2}[p(1-\beta)^2-\beta]^2}} \qquad (l)$$

Here W_1/g is the mass of the main system;

$x_{st} = (Q_0/k_1)$ is the statical deflection of the main mass produced by the force Q_0;

$$m_0 = \sqrt{\frac{k_1g}{W_1}} = \sqrt{\frac{k_2g}{W_2}}; \qquad \beta = \left(\frac{m_0}{m}\right)^2; \qquad p = \frac{W_1}{W_2} = \frac{k_1}{k_2}.$$

From this expression the ratio x_1/x_{st} may be calculated for a definite system and a definite value α of damping, for various values of β, i.e., for various frequencies of the disturbing force, $Q_0 \cos mt$. In figure 107 the results of these calculations for the case $p = 20$, $m_0 = 120$, and $W_1 = 100$ lbs. are given. For $\alpha = .32$, the curve representing the ratio x_1/x_{st} as a function of m/m_0 has two distinct maximums corresponding to two

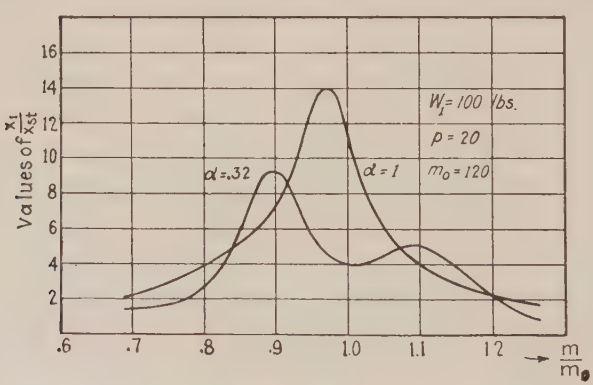


FIG. 107.

resonance frequencies of vibration. For the case $\alpha = 1$, the second resonance peak is damped out. The maximum amplitude in figure 107

* See paper by J. Ormondroyd and J. P. DenHartog, loc. cit.

always occurs at the first peak. Plotting this maximum amplitude as a function of the damping α the curve shown in figure 108 is obtained, from which the *optimum damping* for the system under consideration can be determined. A series of curves like curve 108 has been calculated for

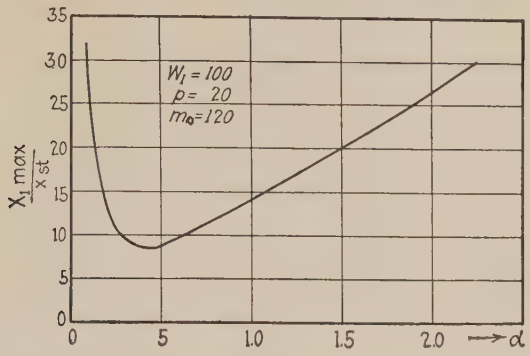


Fig. 108.

various values of p and the corresponding *optimum damping* has been determined. The results of these calculations are given in figure 109 by

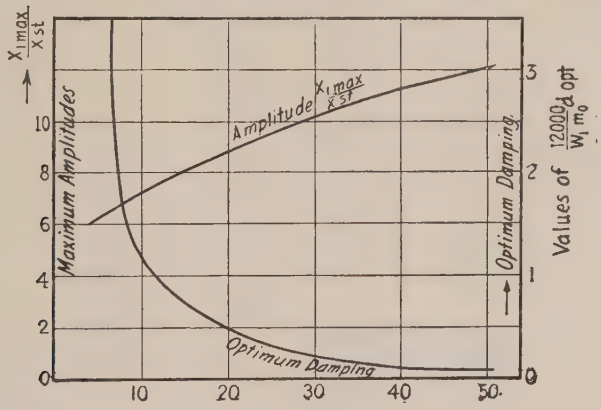


Fig. 109.

a curve, showing the optimum damping α and the maximum amplitude at this damping, both as functions of p. It should be noted that the calculations have been made for $W_1 = 100$ lbs. and $m_0 = 120$, but the results obtained can be used also for other numerical values of these quantities; it is only necessary to take into consideration that equation (l) remains unaltered with changes in W_1, m_0 and α if only

the quantity α/m_0W_1 is kept constant. It is necessary, therefore, in order to adapt the curve (109) to any other value of W_1 or m_0 to read $[(100 \times 120)/m_0W_1]\alpha_{0pt}$ for the ordinates instead of α_{0pt}.

The vibration absorber in the form of a torsional absorber is being applied on several automobile engines. It can be used also to minimize torsional vibrations in crank shafts of Diesel engines. The same principle governs also the famous *Schlingertank* proposed by H. Frahm in 1911 * for stabilizing ships. It consists of two tanks partially filled with water, connected by two pipes (Fig. 110). The upper pipe contains an air throttle. The ship rolling in the water corresponds to the main system in Fig. 103, the impulses of the waves take the place of the disturbing force, and the water surging between the two tanks is the vibration absorber. The damping in the system is regulated by means of the air throttle. The arrangement has proved to be very successful on large passenger steamers. Another type of vibration absorber has been used by H. Frahm for eliminating vibrations in the hull of a ship. A vibratory system analogous to that of a pallograph (see Fig. 11) was attached at the stern of the ship and violent vibrations of the mass of this vibrator produced by vibration of the hull were damped out by a special hydraulic damping arrangement. It was possible in this manner to reduce to a very great extent the vibrations in the hull of the ship produced by unbalanced parts of the engine.

Fig. 110.

* H. Frahm, "Neuartige Schlingertanks zur Abdämpfung von Schiffsrollbewegungen," Jb. d. Schiffbautechn. Ges., Vol. 12, 1911, p. 283.

CHAPTER IV

VIBRATIONS OF ELASTIC BODIES

In considering the vibrations of elastic bodies it will be assumed that the material of the body is homogeneous, isotropic and that it follows Hooke's law. The differential equations of motion established in the previous chapter for a system of particles will also be used here.

In the case of elastic bodies however, instead of several concentrated masses, we have a system consisting of an infinitely large number of particles between which elastic forces are acting. This system requires an infinite large number of coordinates for specifying its position and it therefore has an infinite number of degrees of freedom because any small displacement satisfying the condition of continuity, i.e., a displacement which will not produce cracks in the body, can be taken as a possible or virtual displacement. On this basis it is seen that any elastic body can have an infinite number of natural modes of vibration.

In the case of thin bars and plates the problem of vibration can be considerably simplified. These problems, which are of great importance in many engineering applications, will be discussed in more detail * in the following chapter.

37. Longitudinal Vibrations of Prismatical Bars.—*Differential Equation of Longitudinal Vibrations.*—The following consideration is based on the assumption that during longitudinal vibration of a prismatical bar the cross sections of the bar remain plane and the particles in these cross sections perform only motion in an axial direction of the bar. The longitudinal extensions and compressions which take place during such a vibration of the bar will certainly be accompanied by some lateral deformation, but in the following only those cases will be considered where the length of the longitudinal waves is large in comparison with the cross sectional dimensions of the bar. In these cases the lateral displacements during longitudinal vibration can be neglected without substantial errors.†

* The most complete discussion of the vibration problems of elastic systems can be found in the famous book by Lord Rayleigh "Theory of Sound." See also H. Lamb, "The Dynamical Theory of Sound." A. E. H. Love, "Mathematical Theory of Elasticity," 4 ed. (1927), and Handbuch der Physik, Vol. VI (1928).

† A complete solution of the problem on longitudinal vibrations of a cylindrical bar of circular cross section, in which the lateral displacements are also taken into consideration, was given by L. Pochhammer, Jr. f. Mathem., Vol. 81 (1876), p. 324.

Under these conditions the differential equation of motion of an element of the bar between two adjacent cross sections mn and m_1n_1 (see Fig. 111) may be written in the same manner as for a particle.

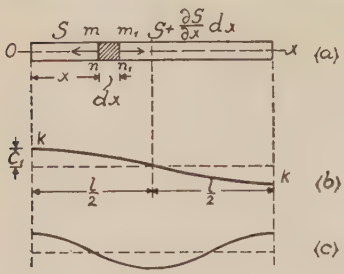


FIG. 111.

Let u = the longitudinal displacement of any cross section mn of the bar during vibration,

e = unit elongation,

E = modulus of elasticity,

A = cross sectional area,

$S = AEe$ = longitudinal tensile force,

γ = weight of the material of the bar per unit volume,

l = the length of the bar.

Then the unit elongation and the tensile force at any cross section mn of the bar will be,

$$e = \frac{\partial u}{\partial x}; \qquad S = AE\frac{\partial u}{\partial x}.$$

For an adjacent cross section the tensile force will be

$$S + dS = AE\left(\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}dx\right).$$

Taking into consideration that the inertia force of the element mnm_1n_1 of the bar is

$$-\frac{A\gamma dx}{g}\frac{\partial^2 u}{\partial t^2},$$

and using the D'Alembert's principle, the following differential equation of motion of the element mnm_1n_1 will be obtained

$$-\frac{A\gamma}{g}\frac{\partial^2 u}{\partial t^2} + AE\frac{\partial^2 u}{\partial x^2} = 0,$$

or

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}, \tag{89}$$

in which *

$$a^2 = \frac{Eg}{\gamma}. \tag{90}$$

Solution by Trigonometric Series.—The displacement u, depending on

* It can be easily shown that a is the velocity of propagation of waves along the bar.

the coordinate x and on the time t, should be such a function of x and t as to satisfy the partial differential equation (89). Particular solutions of this equation can easily be found by taking into consideration first that in the general case any vibration of a system can be resolved into the natural modes of vibration and second, when a system performs one of its natural modes of vibration all points of the system execute a simple harmonic vibration and keep step with one another so that they pass simultaneously through their equilibrium positions. Assume now that the bar performs a natural mode of vibration, the frequency of which is $p/2\pi$, then the solution of eq. (89) should be taken in the following form,

$$u = X(A \cos pt + B \sin pt), \tag{a}$$

in which A and B are arbitrary constants and X a certain function of x alone, determining the shape of the normal mode of vibration under consideration, and called "*normal function.*" This function should be determined in every particular case so as to satisfy the conditions at the ends of the bar. As an example consider now the longitudinal vibrations of a bar with free ends. In this case the tensile force at the ends during vibration should be equal to zero and we obtain the following end conditions (see Fig. 111)

$$\left(\frac{\partial u}{\partial x}\right)_{x=0} = 0; \qquad \left(\frac{\partial u}{\partial x}\right)_{x=l} = 0. \tag{b}$$

Substituting (a) in eq. (89) we obtain

$$-p^2X = a^2\frac{d^2X}{dx^2},$$

from which

$$X = C \cos\frac{px}{a} + D \sin\frac{px}{a}. \tag{c}$$

In order to satisfy the first of the conditions (b) it is necessary to put $D = 0$. The second of the conditions (b) will be satisfied when

$$\sin\frac{pl}{a} = 0. \tag{91}$$

This is the "*frequency equation*" for the case under consideration from which the frequencies of the natural modes of the longitudinal vibrations of a bar with free ends can be calculated. This equation will be satisfied by putting

$$\frac{pl}{a} = i\pi, \tag{d}$$

where i is an integer. Taking $i = 1, 2, 3, \cdots$, the frequencies of the various modes of vibration will be obtained. The frequency of the fundamental type of vibration will be found by putting $i = 1$, then

$$p_1 = \frac{a\pi}{l} = \frac{\pi}{l}\sqrt{\frac{Eg}{\gamma}}. \tag{92}$$

The corresponding period of vibration will be

$$\tau_1 = \frac{2\pi}{p_1} = 2l\sqrt{\frac{\gamma}{Eg}}. \tag{93}$$

The shape of this mode of vibration is represented in Fig. 111, b, by the curve kk, the ordinates of which are equal to

$$X_1 = C_1 \cos\frac{p_1 x}{a} = C_1 \cos\frac{\pi x}{l}.$$

In Fig. 111, c the second mode of vibration is represented in which

$$\frac{p_2 l}{a} = 2\pi; \qquad \text{and} \qquad X_2 = C_2 \cos\frac{2\pi x}{l}.$$

The general form of a particular solution (a) of eq. (89) will be

$$u = \cos\frac{i\pi x}{l}\left(A_i \cos\frac{i\pi at}{l} + B_i \sin\frac{i\pi at}{l}\right). \tag{e}$$

By superimposing such particular solutions any longitudinal vibration of the bar * can be represented in the following form,

$$u = \sum_{i=1,2,3,\ldots}^{i=\infty} \cos\frac{i\pi x}{l}\left(A_i \cos\frac{i\pi at}{l} + B_i \sin\frac{i\pi at}{l}\right). \tag{94}$$

The arbitrary constants A_i, B_i always can be chosen in such a manner as to satisfy any initial conditions.*

Take, for instance, that at the initial moment $t = 0$, the displacements u are given by the equation $(u)_{t=0} = f(x)$ and the initial velocities by the equation $(\dot{u})_{t=0} = f_1(x)$. Substituting $t = 0$ in eq. (94), we obtain

$$f(x) = \sum_{i=1}^{i=\infty} A_i \cos\frac{i\pi x}{l}. \tag{f}$$

By substituting $t = 0$ in the derivative with respect to t of eq. (94), we

* Displacement of the bar as a rigid body is not considered here. An example where this displacement must be taken into consideration will be discussed on p. 207.

obtain

$$f_1(x) = \sum_{i=1}^{i=\infty} \frac{i\pi a}{l} B_i \cos \frac{i\pi x}{l}. \tag{g}$$

The coefficients A_i and B_i in eqs. (f) and (g) can now be calculated, as explained before (see paragraph 12) by using the formulae:

$$A_i = \frac{2}{l} \int_0^l f(x) \cos \frac{i\pi x}{l} dx, \tag{h}$$

$$B_i = \frac{2}{i\pi a} \int_0^l f_1(x) \cos \frac{i\pi x}{l} dx. \tag{k}$$

As an example, consider now the case when a prismatical bar compressed by forces applied at the ends, is suddenly released of this compression at the initial moment $t = 0$. By taking

$$(u)_{t=0} = f(x) = \frac{el}{2} - ex; \qquad f_1(x) = 0,$$

where e denotes the unit compression at the moment $t = 0$, we obtain from eqs. (h) and (k)

$$A_i = \frac{4el}{\pi^2 i^2} \text{ for } i = \text{odd}; \qquad A_i = 0 \text{ for } i = \text{even}; \qquad B_i = 0,$$

and the general solution (94) becomes

$$u = \frac{4el}{\pi^2} \sum_{i=1,3,5,\cdots}^{i=\infty} \frac{\cos \frac{i\pi x}{l} \cos \frac{i\pi a t}{l}}{i^2}.$$

Only odd integers $i = 1, 3, 5, \cdots$ enter in this solution and the vibration is symmetrical about the middle cross section of the bar.

On the general solution (94) representing the vibration of the bar any longitudinal displacement of the bar as a rigid body can be superimposed.

Solution by using Generalized Coordinates.—Taking as generalized coordinates in this case the expressions in the brackets in eq. (e) and using the symbols q_i for these coordinates, we obtain

$$u = \sum_{i=1}^{i=\infty} q_i \cos \frac{i\pi x}{l}. \tag{l}$$

The potential energy of the system consisting in this case of the

energy of tension and compression will be,

$$V = \frac{AE}{2}\int_0^l \left(\frac{\partial u}{\partial x}\right)^2 dx = \frac{AE\pi^2}{2l^2}\int_0^l \left(\sum_{i=1}^{i=\infty} iq_i \sin\frac{i\pi x}{l}\right) dx = \frac{AE\pi^2}{4l}\sum_{i=1}^{i=\infty} i^2 q_i^2 \cdots. \quad (m)$$

In calculating the integral

$$\int_0^l \left(\sum iq_i \sin\frac{i\pi x}{l}\right)^2 dx$$

only the terms containing the squares of the coordinates q_i give integrals different from zero (see paragraph 12).

The kinetic energy at the same time will be,

$$T = \frac{A\gamma}{2g}\int_0^l \left(\frac{\partial u}{\partial t}\right)^2 dx = \frac{A\gamma l}{4g}\sum_{i=1}^{i=\infty} \dot{q}_i^2. \quad (n)$$

Substituting T and V in Lagrange's equations (71) we obtain for each coordinate q_i the following differential equation

$$\ddot{q}_i + \frac{a^2\pi^2 i^2}{l^2} q_i = 0, \quad (p)$$

from which

$$q_i = A_i \cos\frac{i\pi at}{l} + B_i \sin\frac{i\pi at}{l}.$$

This result coincides completely with what was obtained before (see eq. e). We see that the equations (p) contain each only one coordinate q_i. The chosen coordinates are independent of each other and the corresponding vibrations are "principal" modes of vibration of the bar (see p. 124).

The application of generalized coordinates is especially useful in the discussion of forced vibrations. As an example, let us consider here the case of a bar with one end built in and another end free. The solution for this case can be obtained at once from (94). It is only necessary to assume in the previous case that the bar with free ends performs vibrations symmetrical about the middle of the bar. This condition will be satisfied by taking $i = 1, 3, 5 \cdots$ in solution (94). Then the middle section can be considered as fixed and each half of the bar will be exactly in the same condition as a bar with one end fixed and another free. Denoting by l the length of such a bar and putting the origin of coordinates at the fixed

end, the solution for this case will be obtained by substituting $2l$ for l and $\sin i\pi x/2l$ for $\cos i\pi x/l$ in eq. (94). In this manner we have

$$u = \sum_{i=1,3,5,\cdots}^{i=\infty} \sin\frac{i\pi x}{2l}\left(A_i \cos\frac{i\pi at}{2l} + B_i \sin\frac{i\pi at}{2l}\right) \tag{95}$$

Now, if we consider the expressions in the brackets of the above solution as generalized coordinates and use the symbols q_i for them, we obtain,

$$u = \sum_{i=1,3,5,\cdots}^{i=\infty} q_i \sin\frac{i\pi x}{2l}\,. \tag{q}$$

Substituting this in the expressions for the potential and kinetic energy we obtain,

$$V = \frac{AE}{2}\int_0^l \left(\frac{\partial u}{\partial x}\right)^2 dx = \frac{\pi^2 AE}{16l}\sum_{i=1,3,5,\cdots}^{i=\infty} i^2 q_i^2, \tag{96}$$

$$T = \frac{A\gamma}{2g}\int_0^l (\dot{u})^2 dx = \frac{A\gamma l}{4g}\sum_{i=1,3,5,\cdots}^{\infty} \dot{q}_i^2. \tag{97}$$

Lagrange's equation for free vibration corresponding to any coordinate q_i will be as follows:

$$\ddot{q}_i + \frac{a^2 i^2 \pi^2}{4l^2} q_i = 0,$$

from which

$$q_i = A_i \cos\frac{i\pi at}{2l} + B_i \sin\frac{i\pi at}{2l}\,.$$

This coincides with what we had before (see eq. 95).

Forced Vibrations.—If disturbing forces are acting on the bar, Lagrange's equations (72) will be

$$\frac{A\gamma l}{2g}\ddot{q}_i + \frac{\pi^2 i^2 AE}{8l} q_i = Q_i$$

or

$$\ddot{q}_i + \frac{a^2\pi^2 i^2}{4l^2} q_i = \frac{2g}{A\gamma l} Q_i, \tag{r}$$

in which Q_i denotes the generalized force corresponding to the generalized coordinate q_i. In determining this force the general method explained before (see p. 114) will be used. We give an increase δq_i to the coordinate q_i. The corresponding displacement in the bar, as determined from (q), is,

$$\delta u = \delta q_i \sin\frac{i\pi x}{2l}\,.$$

The work done by the disturbing forces on this displacement should now be calculated. This work divided by δq_i represents the generalized force Q_i. Substituting this in eq. (r), the general solution of this equation can easily be obtained, by adding to the free vibrations, obtained above, the vibrations produced by the disturbing force Q_i. This latter vibration is taken usually in the form of a definite integral.* Then,

$$q_i = A_i \cos \frac{i\pi at}{2l} + B_i \sin \frac{i\pi at}{2l} + \frac{4g}{A\gamma a\pi i} \int_0^t Q_i \sin \frac{i\pi a}{2l}(t - t_1)dt_1. \qquad (s)$$

The first two terms in this solution represent a free vibration due to the initial displacement and initial impulse. The third represents the vibration produced by the disturbing force. Substituting solution (s) in eq. (q) the general expression for the vibrations of the bar will be obtained. As an example, the vibration produced by a force $S = f(t)$ acting on the free end of the bar (see Fig. 112) will now be considered. Giving an increase δq_i to the coordinate q_i; the corresponding displacement (see eq. q) will be

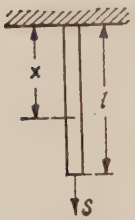


FIG. 112.

$$\delta u = \delta q_i \sin \frac{i\pi x}{2l}.$$

The work produced by the disturbing force on this displacement is

$$S\delta q_i \sin \frac{i\pi}{2}$$

and we obtain

$$Q_i = S \sin \frac{i\pi}{2} = (-1)^{\frac{i-1}{2}} S.$$

Substituting in (s) and taking into consideration only that part of the vibration, produced by the disturbing force, we obtain,

$$q_i = (-1)^{\frac{i-1}{2}} \frac{4g}{A\gamma a\pi i} \int_0^t S \sin \frac{i\pi a}{2l}(t - t_1)dt_1.$$

Substituting in (q) and considering the motion of the lower end of the bar $(x = l)$ we have

$$(u)_{x=l} = \frac{4g}{A\gamma a\pi} \sum_{i=1,3,5,\cdots}^{i=\infty} \frac{1}{i} \int_0^t S \sin \frac{i\pi a}{2l}(t - t_1)dt_1. \qquad (u)$$

* See eq. (24), p. 20.

In any particular case it is only necessary to substitute $S = f(t_1)$ in (u) and perform the integration indicated. Let us take, for instance, the particular case of the vibrations produced in the bar by a constant force suddenly applied at the initial moment $(t = 0)$. Then, from (u), we obtain

$$(u)_{x=l} = \frac{8glS}{A\gamma a^2\pi^2} \sum_{i=1,3,5,\cdots}^{i=\infty} \frac{1}{i^2}\left(1 - \cos\frac{i\pi at}{2l}\right). \tag{98}$$

It is seen that all modes of vibration will be produced in this manner, the periods and frequencies of which are

$$\tau_i = \frac{4l}{ai}; \qquad f_i = \frac{1}{\tau_i} = \frac{ai}{4l}.$$

The maximum deflection will occur when $\cos(i\pi at/2) = -1$. Then

$$(u)_{\substack{x=l \\ t=\frac{2l}{a}}} = \frac{16glS}{A\gamma a^2\pi^2} \sum_{i=1,3,5,\cdots}^{i=\infty} \frac{1}{i^2}$$

or by taking into consideration that

$$a^2 = \frac{Eg}{\gamma} \qquad \text{and} \qquad \sum_{i=1,3,5,\cdots}^{i=\infty} \frac{1}{i^2} = \frac{1}{1} + \frac{1}{3^2} + \frac{1}{5^2} \cdots = \frac{\pi^2}{8}$$

we obtain

$$(u)_{x=l} = \frac{2lS}{AE}.$$

We arrive in this manner at the well known conclusion that a suddenly applied force produces twice as great a deflection as one gradually applied.*

As another example let us consider the longitudinal vibration of a bar with free ends (Fig. 111) produced by a longitudinal force S suddenly applied at the end $x = l$. Superposing on the vibration of the bar given by eq. (l) a displacement q_1 of the bar as a rigid body the displacement u can be represented in the following form:

$$u = q_1 + q_1 \cos\frac{\pi x}{l} + q_2 \cos\frac{2\pi x}{l} + q_3 \cos\frac{3\pi x}{l} + \cdots. \tag{v}$$

The expressions for potential and kinetic energy, from (m) and (n) will be

$$V = \frac{AE\pi^2}{4l} \sum_{i=1}^{i=\infty} i^2 q_i^2; \qquad T = \frac{A\gamma l}{2g}\dot{q}_1^2 + \frac{A\gamma l}{4g}\sum_{i=1}^{i=\infty} \dot{q}_i^2$$

* For a more detailed discussion of this subject see the next paragraph (p. 214).

and the equations of motion become

$$\begin{aligned} &\frac{A\gamma l}{g}\ddot{q}_0 = Q_0, \\ &\cdots\cdots\cdots \\ &\cdots\cdots\cdots \\ &\frac{A\gamma l}{2g}\ddot{q}_i + \frac{AE\pi^2 i^2}{2l}q_i = Q_i, \\ &\cdots\cdots\cdots \end{aligned} \tag{w}$$

By using the same method as before (see p. 206) it can be shown that in this case

$$Q_0 = S \qquad \text{and} \qquad Q_i = (-1)^i S.$$

Then assuming that the initial velocities and the initial displacements are equal to zero, we obtain, from eqs. (w):

$$q_0 = \frac{gSt^2}{2A\gamma l},$$

$$q_i = (-1)^i \frac{2g}{A\pi a\gamma i}\int_0^t S\sin\frac{i\pi a}{l}(t - t_1)dt_1 = \frac{(-1)^i 2glS}{A\pi^2 i^2 \gamma a^2}\left(1 - \cos\frac{i\pi at}{l}\right).$$

Substituting in eq. (v), the following solution for the displacements produced by a suddenly applied force S will be obtained

$$u = \frac{gSt^2}{2A\gamma l} + \frac{2glS}{A\pi^2 a^2\gamma}\sum_{i=1}^{i=\infty}\frac{(-1)^i}{i^2}\cos\frac{i\pi x}{l}\left(1 - \cos\frac{i\pi at}{l}\right).$$

The first term on the right side represents the displacement calculated as for a rigid body. To this displacement, vibrations of a bar with free ends are added. Using the notations $\delta = (Sl/AE)$ for the elongation of the bar uniformly stretched by the force S, and $\tau = (2l/a)$ for the period of the fundamental vibration, the displacement of the end $x = l$ of the bar will be

$$(u)_{x=l} = \frac{2\delta t^2}{\tau^2} + \frac{2\delta}{\pi^2}\sum_{i=1}^{i=\infty}\frac{1}{i^2}\left(1 - \cos\frac{2i\pi t}{\tau}\right).$$

The maximum displacement, due to vibration, will be obtained when $t = (\tau/2)$. Then

$$(u)_{x=l} = \frac{\delta}{2} + \frac{4\delta}{\pi^2}\left(\frac{1}{1} + \frac{1}{9} + \frac{1}{25} + \cdots\right) = \frac{\delta}{2} + \frac{4\delta}{\pi^2}\frac{\pi^2}{8} = \delta.$$

An analogous problem is encountered in investigating the vibrations produced during the lifting of a long drill stem as used in deep oil wells.

38. Vibration of a Bar with a Load at the End.—*Natural Vibrations.*—The problem of the vibration of a bar with a load at the end may have a practical application not only in the case of prismatical bars but also when the load is supported by a helical spring as in the case of an indicator spring (see p. 16.) If the mass of the bar or of the spring be small in comparison with the mass of the load at the end it can be neglected and the problem will be reduced to that of a system with one degree of freedom (see Fig. 1). In the following the effect of the mass of the bar will be considered in detail. Denoting the longitudinal displacements from the position of equilibrium by u and using the differential equation (89) of the longitudinal vibrations developed in the previous paragraph, we obtain

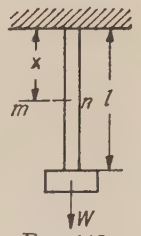


FIG. 113.

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \qquad (89)^1$$

where

$$a^2 = \frac{Eg}{\gamma}$$

for a prismatical bar, and

$$a^2 = \frac{klg}{q}$$

for a helical spring. In this latter case k = the spring constant, this being the load necessary to produce a total elongation of the spring equal to unity. l = the length of the spring and q = the weight of the spring per unit length. The end conditions will be as follows.

At the built-in end the displacement should be zero during vibration and we obtain

$$(u)_{x=0} = 0. \qquad (a)$$

At the lower end, at which the load is attached, the tensile force in the bar must be equal to the inertia force of the oscillating load W and we have *

$$AE\left(\frac{\partial u}{\partial x}\right)_{x=l} = -\frac{W}{g}\left(\frac{\partial^2 u}{\partial t^2}\right)_{x=l}. \qquad (b)$$

Assuming that the system performs one of the principal modes of vibration we obtain

$$u = X(A \cos pt + B \sin pt), \qquad (c)$$

in which X is a *normal function* of x alone, determining the shape of the mode of vibration.

Substituting (c) in eq. $(89)^1$ we obtain

$$a^2 \frac{d^2X}{dx^2} + p^2 X = 0,$$

from which

$$X = C \cos \frac{px}{a} + D \sin \frac{px}{a}, \qquad (d)$$

where C and D are arbitrary constants of integration.

In order to satisfy condition (a) we have to take $C = 0$ in solution (d). From condition (b) we obtain

$$AE \frac{p}{a} \cos \frac{pl}{a} = \frac{W}{g} p^2 \sin \frac{pl}{a}. \qquad (b)^1$$

* The constant load W, being in equilibrium with the uniform tension of the bar in its position of equilibrium, will not affect the end condition.

Let $\alpha = A\gamma l/W$ = ratio of the weight of the bar to the weight of the load W and $\beta = pl/a$. Then eq. (b)[1] becomes

$$\alpha = \beta \tan \beta. \tag{99}$$

This is the *frequency equation* for the case under consideration, the roots of which can be easily obtained graphically, provided the ratio α be known. The fundamental type of vibration is usually the most important in practical applications and the values β_1 of the smallest root of eq. (99) for various values of α are given in the table below.

$\alpha =$	.01	.10	.30	.50	.70	.90	1.00	1.50	2.00	3.00	4.00	5.00	10.0	20.0	100.0	∞
$\beta_1 =$	.10	.32	.52	.65	.75	.82	.86	.98	1.08	1.20	1.27	1.32	1.42	1.52	1.568	$\pi/2$

If the weight of the bar is small in comparison with the load W, the quantity α and the root β_1 will be small and equation (99) can be simplified by putting $\tan \beta = \beta$, then

$$\beta^2 = \alpha = \frac{A\gamma l}{W}$$

and we obtain

$$\beta = \frac{pl}{a} = \sqrt{\frac{A\gamma l}{W}} \tag{e}$$

$$\text{and } p = \frac{a}{l}\sqrt{\frac{A\gamma l}{W}} = \sqrt{\frac{g}{\delta_{st}}}, \tag{f}$$

where $\delta_{st} = Wl/AE$ represents the statical elongation of the bar under the action of the load W.

This result coincides with the one obtained before for a system with one degree of freedom (see eq. 3, p. 2). A better approximation will be obtained by substituting $\tan \beta = \beta + \beta^3/3$ in eq. (99). Then

$$\beta(\beta + \beta^3/3) = \alpha$$

or

$$\beta = \sqrt{\frac{\alpha}{1 + \beta^2/3}}. \tag{g}$$

Substituting the first approximation (e) for β in the right side of this equation, we obtain

$$\beta = \sqrt{\frac{\alpha}{1 + \alpha/3}}\,(h) \quad \text{and} \quad p = \sqrt{\frac{g}{\delta_{st}(1 + \alpha/3)}} \tag{h}$$

Comparing (h) with (f) it can be concluded that the better approximation is obtained by adding one third of the weight of the bar to the weight W of the load. This is the well-known approximate solution obtained before by using Rayleigh's method (see p. 60).

Comparing the approximate solution (h) with the data of the table above it can be concluded that for $\alpha = 1$ the error arising from the use of the approximate formula is less than 1% and in all cases when the weight of the bar is less than the weight of the load it is satisfactory for practical applications.

Assuming that for a given α the consecutive roots $\beta_1, \beta_2, \beta_3, \cdots$ of the frequency equation (99) are calculated, and substituting $\beta_i a/l$ for p in solution (c) we obtain,

$$u_i = \sin \frac{\beta_i x}{l}\left(A_i \cos \frac{\beta_i at}{l} + B_i \sin \frac{\beta_i at}{l}\right).$$

This solution represents a principal mode of vibration of the order i of our system. By superimposing such vibrations any vibration of the bar with a load at the end can be obtained in the form of a series,

$$u = \sum_{i=1}^{i=\infty} \sin \frac{\beta_i x}{l} \left(A_i \cos \frac{\beta_i a t}{l} + B_i \sin \frac{\beta_i a t}{l} \right), \tag{k}$$

the arbitrary constants A_i and B_i of which should be determined from the initial conditions.

Assume, for instance, that the bar is at rest under the action of a tensile force S applied at the lower end and that at the initial moment $t = 0$ this force is suddenly removed. For this case all the coefficients B_i in eq. (k) should be taken equal to zero because the initial velocities are zero. The coefficients A_i should be determined in such a manner as to represent the initial configuration of the system. From the uniform extension of the bar at the initial moment we obtain

$$(u)_{t=0} = \frac{Sx}{AE}.$$

Equation (k), for $t = 0$, yields

$$(u)_{t=0} = \sum_{i=1}^{i=\infty} A_i \sin \frac{\beta_i x}{l}.$$

The coefficients A_i should be determined in such a manner as to satisfy the equation

$$\sum_{i=1}^{i=\infty} A_i \sin \frac{\beta_i x}{l} = \frac{Sx}{AE}. \tag{l}$$

In determining these coefficients we proceed exactly as was explained in paragraph 12. In order to obtain any coefficient A_i both sides of the above equation should be multiplied with $\sin (\beta_i x/l) dx$ and integrated from $x = 0$ to $x = l$. By simple calculations we obtain

$$\int_0^l \sin^2 \frac{\beta_i x}{l} dx = \frac{l}{2} \left(1 - \frac{\sin 2\beta_i}{2\beta_i} \right),$$

$$\frac{S}{AE} \int_0^l x \sin \frac{\beta_i x}{l} dx = \frac{Sl^2}{AE} \left(- \frac{\cos \beta_i}{\beta_i} + \frac{\sin \beta_i}{\beta_i^2} \right)$$

and also, by taking into consideration eq. 99, for every integer $m \neq i$

$$\int_0^l \sin \frac{\beta_i x}{l} \sin \frac{\beta_m x}{l} dx = - \frac{W}{A\gamma} \sin \beta_i \sin \beta_m = - \frac{l}{\alpha} \sin \beta_i \sin \beta_m.$$

Then, from eq. (l)

$$\int_0^l \sin \frac{\beta_i x}{l} \sum_{i=1}^{i=\infty} A_i \sin \frac{\beta_i x}{l} dx = \frac{S}{AE} \int_0^l x \sin \frac{\beta_i x}{l} dx$$

or

$$A_i \frac{l}{2} \left(1 - \frac{\sin 2\beta_i}{2\beta_i} \right) - \frac{l}{\alpha} \sin \beta_i \sum_{m=1,2,3,\cdots i-1, i+1, \cdots}^{m=\infty} A_m \sin \beta_m = \frac{Sl^2}{AE} \left(- \frac{\cos \beta_i}{\beta_i} + \frac{\sin \beta_i}{\beta_i^2} \right).$$

Remembering that, from eq. (k),

$$\sum_{i=1,2,3,\cdots (i-1),(i+1)\cdots}^{i=\infty} A_m \sin \beta_m = (u)_{\substack{x=l \\ t=0}} - A_i \sin \beta_i = \frac{Sl}{AE} - A_i \sin \beta_i$$

we obtain

$$A_i \frac{l}{2}\left(1 - \frac{\sin 2\beta_i}{2\beta_i}\right) - \frac{l}{\alpha}\sin\beta_i\left(\frac{Sl}{AE} - A_i \sin\beta_i\right) = \frac{Sl^2}{AE}\left(-\frac{\cos\beta_i}{\beta_i} + \frac{\sin\beta_i}{\beta_i^2}\right),$$

from which, by taking into consideration that (from eq. 99)

$$\frac{l}{\alpha}\sin\beta_i = \frac{l\cos\beta_i}{\beta_i},$$

we obtain

$$A_i = \frac{4Sl\sin\beta_i}{AE\beta_i(2\beta_i + \sin 2\beta_i)},$$

the initial displacement will be

$$(u)_{t=0} = \frac{Sx}{AE} = \frac{4Sl}{AE}\sum_{i=1}^{i=\infty}\frac{\sin\beta_i \sin\frac{\beta_i x}{l}}{\beta_i(2\beta_i + \sin 2\beta_i)}, \tag{100}$$

and the vibration of the bar will be represented in this case by the following series:

$$u = \frac{4Sl}{AE}\sum_{i=1}^{i=\infty}\frac{\sin\beta_i \sin\frac{\beta_i x}{l}\cos\frac{\beta_i at}{l}}{\beta_i(2\beta_i + \sin 2\beta_i)}. \tag{101}$$

Forced Vibrations.—In the following the forced vibrations of the system will be considered by taking the expressions in the brackets of eq. (k) for generalized coordinates. Then

$$u = \sum_{i=1}^{i=\infty} q_i \sin\frac{\beta_i x}{l}. \tag{m}$$

The potential energy of the system will be,

$$V = \frac{AE}{2}\int_0^l\left(\frac{\partial u}{\partial x}\right)^2 dx = \frac{AE}{2l^2}\int_0^l\left(\sum_{i=1}^{i=\infty}\beta_i q_i \cos\frac{\beta_i x}{l}\right)^2 dx.$$

It can be shown by simple calculations that, in virtue of eq. (99),

$$\int_0^l \cos\frac{\beta_n x}{l}\cos\frac{\beta_m x}{l}\,dx = 0 \qquad \text{when} \qquad m \neq n^*$$

and

$$\int_0^l \cos^2\frac{\beta_m x}{l}\,dx = \frac{l}{2}\left(1 + \frac{\sin 2\beta_m}{2\beta_m}\right).$$

Substituting in the above expression for V we obtain,

$$V = \frac{AE}{4l}\sum_{i=1}^{i=\infty}\beta_i^2 q_i^2\left(1 + \frac{\sin 2\beta_i}{2\beta_i}\right). \tag{n}$$

The kinetic energy of the system will consist of two parts, the kinetic energy of the vibrating rod and the kinetic energy of the load at the end of the rod, and we obtain

$$T = \frac{A\gamma}{2g}\int_0^l (\dot{u})^2 dx + \frac{W}{2g}(\dot{u})^2_{x=l}.$$

* The same can be concluded also from the fact that the coordinates $q_1, q_2, \cdots$ are principal coordinates, hence the potential and kinetic energies should contain only squares of these coordinates.

Substituting (m) for the displacement u and performing the integrations:

$$\int_0^l \sin^2 \frac{\beta_m x}{l}\, dx = \frac{l}{2}\left(1 - \frac{\sin 2\beta_m}{2\beta_m}\right)$$

$$\int_0^l \sin \frac{\beta_m x}{l} \sin \frac{\beta_n x}{l}\, dx = -\frac{W}{A\gamma} \sin \beta_m \sin \beta_n, \qquad \text{where} \quad m \neq n.$$

We obtain

$$T = \frac{A\gamma l}{4g} \sum_{i=1}^{i=\infty} \dot{q}_i^2 \left(1 - \frac{\sin 2\beta_i}{2\beta_i}\right) + \frac{W}{2g} \sum_{i=1}^{i=\infty} \dot{q}_i^2 \sin^2 \beta_i.$$

Now, from eq. (99), we have

$$\alpha = \frac{A\gamma l}{W} = \beta_i \tan \beta_i$$

or

$$W = \frac{A\gamma l}{\beta_i \tan \beta_i}.$$

Substituting in the above expression for the kinetic energy we obtain,

$$T = \frac{A\gamma l}{4g} \sum_{i=1}^{i=\infty} \dot{q}_i^2 \left(1 + \frac{\sin 2\beta_i}{2\beta_i}\right). \tag{o}$$

It is seen that the expressions (n) and (o) for the potential and kinetic energy contain only squares of q_i and $\dot{q}_i$. The products of these quantities disappear because the terms of the series (k) and (l) are the *principal or natural modes* of vibration of the system under consideration and the coordinates q_i are the *principal coordinates* (see p. 124). Substituting (n) and (o) in Lagrange's equation (72) the following equation for any coordinate q_i will be obtained.

$$\frac{A\gamma l}{2g}\left(1 + \frac{\sin 2\beta_i}{2\beta_i}\right)\ddot{q}_i + \frac{AE}{2l}\beta_i^2\left(1 + \frac{\sin 2\beta_i}{2\beta_i}\right) q_i = Q_i, \tag{p}$$

in which Q_i denotes the generalized force corresponding to the generalized coordinate q_i.

Considering only vibrations produced by a disturbing force and neglecting the free vibrations due to initial displacements and initial impulses, the solution of eq. (p) will be *

$$q_i = \frac{2g}{A\gamma l} \cdot \frac{l}{a\beta_i} \frac{2\beta_i}{2\beta_i + \sin 2\beta_i} \int_0^t Q_i \sin \frac{a\beta_i}{l}(t - t_1)dt_1,$$

where, as before,

$$a = \sqrt{\frac{Eg}{\gamma}}.$$

Substituting this into (m) the following general solution of the problem will be obtained:

$$u = \frac{4g}{Aa\gamma} \sum_{i=1}^{i=\infty} \frac{\sin \frac{\beta_i x}{l}}{2\beta_i + \sin 2\beta_i} \int_0^t Q_i \sin \frac{a\beta_i}{l}(t - t_1)dt_1. \tag{102}$$

In any particular case the corresponding value of Q_i should be substituted in this solution. By putting $x = l$ the displacements of the load W during vibration will be obtained.

* See eq. 24, p. 20.

Force Suddenly Applied.—Consider, as an example, the vibration produced by a constant force S suddenly applied at the lower end of the bar. The generalized force Q_i corresponding to any coordinate q_i in this case (see p. 114) will be

$$Q_i = S \sin \beta_i.$$

Substituting in eq. (102) we obtain for the displacements of the load W the following expression:

$$(u)_{x=l} = \frac{4gSl}{Aa^2\gamma} \sum_{i=1}^{i=\infty} \frac{\sin^2 \beta_i}{\beta_i(2\beta_i + \sin 2\beta_i)} \left(1 - \cos \frac{a\beta_i t}{l}\right) \cdots. \tag{103}$$

Consider now the particular case when the load W at the end of the bar diminishes to zero and the conditions approach those considered in the previous paragraph. In such a case α in eq. (99) becomes infinitely large and the roots of this transcendental equation will be

$$\beta_i = \frac{(2i - 1)\pi}{2}.$$

Substituting in eq. (103) the same result as in the previous paragraph (see eq. 98, p. 207) will be obtained.

A second extreme case is when the load W is very large in comparison with the weight of the rod and α in eq. (99) approaches zero. The roots of this equation then approach the values:

$$\beta_i = (i - 1)\pi.$$

All terms in the series (103) except the first term, tend towards zero and the system approaches the case of one degree of freedom. The displacement of the lower end of the rod will be given in this case by the first term of (103) and will be

$$(u)_{x=l} = \frac{4gSl}{Aa^2\gamma} \frac{\sin^2 \beta_1}{\beta_1(2\beta_1 + \sin 2\beta_1)} \left(1 - \cos \frac{a\beta_1 t}{l}\right)$$

or by putting $\sin \beta_1 = \beta_1$ and $\sin 2\beta_1 = 2\beta_1$, we obtain

$$(u)_{x=l} = \frac{gSl}{Aa^2\gamma} \left(1 - \cos \frac{a\beta_1 t}{l}\right).$$

This becomes a maximum when

$$\cos \frac{a\beta_1 t}{l} = -1,$$

then

$$(u)_{\max} = \frac{2gSl}{Aa^2\gamma} = \frac{2Sl}{AE}.$$

This shows that the maximum displacement produced by a suddenly applied force is twice as great as the static elongation produced by the same force.

This conclusion also holds for the case when $W = 0$ (see p. 207) but it will not be true in the general case given by eq. 103. To prove this it is necessary to observe that in the two particular cases mentioned above, the system at the end of a half period of the fundamental mode of vibration will be in a condition of instantaneous rest. At this moment the kinetic energy becomes equal to zero and the work done by the suddenly applied constant force is completely transformed into potential energy of deformation and it can be concluded from a statical consideration that the displacement of the point of application of the force should be twice as great as in the equilibrium configuration.

In the general case represented by eq. (103) the roots of eq. (99) are incommensurable and the system never passes into a configuration in which the energy is purely potential. Part of the energy always remains in the form of kinetic energy and the displacement of the point of application of the force will be less than twice that in the equilibrium configuration.

Comparison with Static Deflection.—The method of generalized coordinates, applied above, is especially useful for comparing the displacements of a system during vibration and the statical displacements which would be produced in the system if the disturbing forces vary very slowly. Such comparisons are necessary, for instance, in the study of steam and gas engine indicator diagrams, and of various devices used in recording gas pressures during explosions. The case of an indicator is represented by the scheme shown in Fig. 113. Assume that a pulsating force $S \sin pt$ is applied to the load W, representing the reduced mass of the piston (see p. 17). In order to find the generalized force in this case, the expression (m) for the displacements will be used. Giving to a coordinate q_i an increase δq_i the corresponding displacement in the bar will be

$$\delta q_i \sin \frac{\beta_i x}{l}$$

and the work done by the pulsating load $S \sin pt$ during this displacement will be

$$S \sin pt \sin \beta_i \delta q_i.$$

Hence the generalized force

$$Q_i = S \sin pt \sin \beta_i.$$

Substituting this in solution (102) and performing the integration we obtain

$$(u)_{x=l} = \frac{2gS}{A\gamma l} \sum_{i=1}^{i=\infty} \frac{\sin^2 \beta_i \left(\sin pt - \frac{pl}{a\beta_i} \sin \frac{a\beta_i t}{l} \right)}{\left(1 + \frac{\sin 2\beta_i}{2\beta_i} \right) \left(\frac{a^2 \beta_i^2}{l^2} - p^2 \right)}. \tag{q}$$

It is seen that the vibration consists of two parts: (1) forced vibrations porportional to *sin pt* having the same period as the disturbing force and (2) free vibrations proportional to $\sin a\beta_i t/l$. When the frequency of the disturbing force approaches one of the natural frequencies of vibration, p approaches the value $a\beta_i/l$ for this mode of vibration and a condition of resonance takes place. The amplitude of vibration of the corresponding term in the series (q) will then increase indefinitely, as was explained before (see pp. 9 and 127). In order to approach the static condition the quantity p should be considered as small in comparison with $a\beta_i/l$ in the series (q). Neglecting then the terms having $pl/a\beta_i$ as a factor, we obtain, for a very slow variation of the pulsating load,

$$(u)_{x=l} = \frac{4lS \sin pt}{AE} \sum_{i=1}^{i=\infty} \frac{\sin^2 \beta_i}{\beta_i(2\beta_i + \sin 2\beta_i)}, \tag{r}$$

which represents the static elongation of the bar (see eq. 100). By comparing the series (r) and (q) the difference between static and dynamic deflections can be established.* It is seen that a satisfactory record of steam or gas pressure can be obtained only if the frequency of the fundamental mode of vibration of the indicator is high in comparison with the frequency of the pulsating force.

* Damping effect is neglected in this consideration.

39. Torsional Vibration of Circular Shafts.—*Free Vibration.*—In our previous discussions (see pp. 5 and 130) the mass of the shaft was either neglected or considered small in comparison with the rotating masses attached to the shaft. In the following a more complete theory of the torsional vibrations of a circular shaft with two discs at the ends is given * on the basis of which the accuracy of our previous solution is discussed. It is assumed in the following discussion that the circular cross sections of the shaft during torsional vibration remain plane and the radii of these cross sections remain straight.† Let

$GI_p = C =$ torsional rigidity of shaft,

$\gamma =$ weight per unit volume of shaft,

$\theta =$ angle of twist at any arbitrary cross section mn (see Fig. 111) during torsional vibration,

$I_1, I_2 =$ moments of inertia of the discs at the ends of the shaft about the shaft axis.

Considering an element of the shaft between two adjacent cross sections mn and m_1n_1 the twisting moments at these cross sections will be

$$GI_p \frac{\partial \theta}{\partial x} \qquad \text{and} \qquad GI_p \left(\frac{\partial \theta}{\partial x} + \frac{\partial^2 \theta}{\partial x^2} dx \right).$$

The differential equation of rotatory motion of the elemental disc mnm_1n_1 (see Fig. 111) during torsional vibration will be

$$\frac{\gamma I_p}{g} \frac{\partial^2 \theta}{\partial t^2} = GI_p \frac{\partial^2 \theta}{\partial x^2}$$

or by using the notation

$$\frac{Gg}{\gamma} = a^2 \tag{104}$$

we obtain

$$\frac{\partial^2 \theta}{\partial t^2} = a^2 \frac{\partial^2 \theta}{\partial x^2}. \tag{105}$$

This equation is identical with the equation (89) obtained above for the longitudinal vibration and the previous results can be used in various particular cases. For instance, in the case of a shaft with free ends the

* See writer's paper in the Bulletin of the Polytechnical Institute in S. Petersburg, 1905, and also his paper "Ueber die Erzwungenen Schwingungen von Prismatischen Stäben," Z. f. Math. u. Phys., Vol. 59 (1911).

† A more complete theory can be found in L. Pochhammer's paper, mentioned above (p. 199).

frequency equation will be identical with eq. (91) and the general solution will be (see eq. 94).

$$\theta = \sum_{i=1}^{i=\infty} \cos\frac{i\pi x}{l}\left(A_i \cos\frac{i\pi at}{l} + B_i \sin\frac{i\pi at}{l}\right). \tag{106}$$

In the case of a shaft with discs at the ends the problem becomes more complicated and the end conditions must be considered. From the condition that the twisting of the shaft at the ends is produced by the inertia forces of the discs we obtain

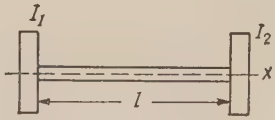


FIG. 114.

$$I_1\left(\frac{\partial^2\theta}{\partial t^2}\right)_{x=0} = GI_p\left(\frac{\partial\theta}{\partial x}\right)_{x=0}, \tag{a}$$

$$I_2\left(\frac{\partial^2\theta}{\partial t^2}\right)_{x=l} = -GI_p\left(\frac{\partial\theta}{\partial x}\right)_{x=l}. \tag{b}$$

Assume that the shaft performs one of the normal modes of vibration, then it can be written:

$$\theta = X(A \cos pt + B \sin pt), \tag{c}$$

where X is a function of x alone, determining the shape of the mode of vibration under consideration.

Substituting (*c*) in eq. (105) we obtain

$$a^2\frac{d^2X}{dx^2} + p^2X = 0,$$

from which

$$X = C \cos\frac{px}{a} + D \sin\frac{px}{a}. \tag{d}$$

The arbitrary constants C and D should be determined in such a manner as to satisfy the end conditions. Substituting (*d*) in eqs. (*a*) and (*b*) we obtain

$$-Cp^2I_1 = D\frac{p}{a}GI_p, \tag{e}$$

$$p^2\left(C \cos\frac{pl}{a} + D \sin\frac{pl}{a}\right)I_2 = \frac{p}{a}GI_p\left(-C \sin\frac{pl}{a} + D \cos\frac{pl}{a}\right).$$

Eliminating the arbitrary constants C and D the following *frequency equation* will be obtained,

$$p^2\left(\cos\frac{pl}{a} - \frac{paI_1}{GI_p}\sin\frac{pl}{a}\right)I_2 = -\frac{p}{a}GI_p\left(\sin\frac{pl}{a} + \frac{paI_1}{GI_p}\cos\frac{pl}{a}\right). \tag{f}$$

Letting

$$\frac{pl}{a} = \beta; \qquad \frac{I_1 g}{\gamma l I_p} = \frac{I_1}{I_0} = m; \qquad \frac{I_2}{I_0} = n, \tag{g}$$

where $I_0 = (\gamma l I_p/g)$ is the moment of inertia of the shaft about its axis, we obtain, from eq. (f) the frequency equation in the following form

$$\beta n(1 - m\beta \tan \beta) = -(\tan \beta + m\beta)$$

or

$$\tan \beta = \frac{(m + n)\beta}{mn\beta^2 - 1}. \tag{107}$$

Let

$$\beta_1, \beta_2, \beta_3, \cdots$$

be the consecutive roots of this transcendental equation, then the corresponding normal functions, from (d) and (e) will be

$$X_i = C_i\left(\cos\frac{\beta_i x}{l} - m\beta_i \sin\frac{\beta_i x}{l}\right)$$

and we obtain for the general solution in this case

$$\theta = \sum_{i=1}^{i=\infty}\left(\cos\frac{\beta_i x}{l} - m\beta_i \sin\frac{\beta_i x}{l}\right)\left(A_i \cos\frac{\beta_i at}{l} + B_i \sin\frac{\beta_i at}{l}\right). \tag{108}$$

If the moments of inertia I_1 and I_2 of the discs are small in comparison with the moment of inertia I_0 of the shaft, the quantities m and n in eq. 107 become small, the consecutive roots of this equation will approach the values π, 2π, $\cdots$ and the general solution (108) approaches the solution (106) given above for a shaft with free ends.

Consider now another extreme case, more interesting from a practical standpoint, when I_1 and I_2 are large in comparison with I_0; the quantities m and n will then be large numbers. In this case unity can be neglected in comparison with $mn\beta^2$ in the denominator on the right side of eq. 107 and, instead of eq. (107), we obtain

$$\beta \tan \beta = (1/m + 1/n). \tag{109}$$

This equation is of the same form as eq. (99) (see p. 210) for longitudinal vibrations. The right side of this equation is a small quantity and an approximate solution for the first root will be obtained by substituting $\tan \beta_1 = \beta_1$. Then

$$\beta_1 = \sqrt{1/m + 1/n}. \tag{h}$$

The period of the corresponding mode of vibration, from eq. 108, will be

$$\tau_1 = 2\pi : \frac{\beta_1 a}{l} = \frac{2\pi l}{\beta_1 a}$$

or, by using equations 104, (g) and (h) we obtain

$$\tau_1 = 2\pi\sqrt{\frac{lI_1I_2}{GI_p(I_1 + I_2)}}. \qquad (15)^1$$

This result coincides with eq. 15 (see p. 8) obtained by considering the system as having one degree of freedom and neglecting the mass of the shaft.

The approximate values of the consecutive roots of eq. (109) will be,

$$\beta_2 = \pi + 1/\pi(1/m + 1/n); \qquad \beta_3 = 2\pi + 1/2\pi(1/m + 1/n); \cdots.$$

It is seen that all these roots are large in comparison with β_1 and the frequencies of the corresponding modes of vibration will be very high in comparison with the frequency of fundamental type of vibration.

In order to get a closer approximation for the first root of eq. (107), we substitute $\tan \beta_1 = \beta_1 + 1/3\beta_1^3$, then

$$\beta_1 + \frac{1}{3}\beta_1^3 = \frac{\beta_1(m + n)}{mn\beta_1^2 - 1}$$

or

$$\beta_1^2 = \frac{m + n}{\left(mn - \frac{1}{\beta_1^2}\right)\left(1 + \frac{1}{3}\beta_1^2\right)}.$$

Substituting in the right side of this equation the value of β_1 from eq. (h) and neglecting small quantities of higher order, we obtain

$$\beta_1 = \sqrt{\frac{1}{m} + \frac{1}{n} - \frac{1}{3}\left(\frac{1}{n^2} + \frac{1}{m^2} - \frac{1}{mn}\right)},$$

and the corresponding frequency of the fundamental vibration will be

$$f_1 = \frac{\beta_1 a}{2\pi l} = \frac{a}{2\pi l}\sqrt{\frac{1}{m} + \frac{1}{n} - \frac{1}{3}\left(\frac{1}{n^2} + \frac{1}{m^2} - \frac{1}{mn}\right)}. \qquad (110)$$

The same result will be obtained if in the first approximation for the frequency

$$f_1^1 = \frac{1}{2\pi}\sqrt{\frac{GI_p(I_1 + I_2)}{lI_1I_2}}$$

as obtained from eq. $(15)^1$ we substitute

$$I_1 + \frac{1}{3}I_0\frac{I_2}{I_1 + I_2} \quad \text{and} \quad I_2 + \frac{1}{3}I_0\frac{I_1}{I_1 + I_2} \text{ for } I_1 \text{ and } I_2.$$

This means that the second approximation (110) coincides with the result which would have been obtained by the Rayleigh method (see paragraph 14, p. 55). According to this method one third of the moment of inertia of the part of the shaft between the disc and the nodal cross section should be added to the moment of inertia of each disc. This approximation is always sufficient in practical applications for calculating the frequency of the fundamental mode of vibration.*

Forced Vibration.—In studying forced torsional vibrations generalized coordinates again are very useful. Considering the brackets containing t in the general solution (108) as such coordinates, we obtain

$$\theta = \sum_{i=1}^{i=\infty} q_i \left(\cos \frac{\beta_i x}{l} - m\beta_i \sin \frac{\beta_i x}{l} \right), \tag{108^1}$$

in which β_i are consecutive roots of eq. (107).

The potential energy of the system will be

$$V = \frac{GI_p}{2} \int_0^l \left(\frac{\partial \theta}{\partial x} \right)^2 dx = \frac{GI_p}{2} \int_0^l \left\{ \sum_{i=1}^{i=\infty} \frac{\beta_i}{l} q_i \left(\sin \frac{\beta_i x}{l} + m\beta_i \cos \frac{\beta_i x}{l} \right) \right\}^2 dx$$

$$= \frac{GI_p}{8l} \sum_{i=1}^{i=\infty} A_i \beta_i q_i^2, \tag{111}$$

where

$$A_i = 2\beta_i(1 + m^2\beta_i^2) - \sin 2\beta_i + m^2\beta_i^2 \sin 2\beta_i + 2\beta_i m(1 - \cos 2\beta_i). \tag{k}$$

The terms containing products of the coordinates in expression (111) disappear in the process of integration in virtue of eq. (107). Such a result should be expected if we remember that our generalized coordinates are principal or normal coordinates of the system.

The kinetic energy of the system consists of the energy of the vibrating shaft and of the energies of the two oscillating discs:

$$T = \frac{\gamma I_p}{2g} \int_0^l \dot{\theta}^2 dx + \frac{1}{2} I_1 \dot{\theta}^2{}_{x=0} + \frac{1}{2} I_2 \dot{\theta}^2{}_{x=l}$$

or, substituting (108)^1 for θ we obtain

$$T = \frac{I_0}{8} \sum_{i=1}^{i=\infty} A_i \frac{1}{\beta_i} \dot{q}_i^2, \tag{112}$$

in which A_i is given by eq. (k).

* A graphical method for determining the natural frequencies of torsional vibration of shafts with discs has been developed by F. M. Lewis, see papers: "Torsional Vibrations of Irregular Shafts," Journal Am. Soc. of Naval Engs. Nov. 1919, p. 857 and "Critical Speeds of Torsional Vibration," Journal Soc. Automotive Engs., Nov. 1920, p. 413.

By using equations (111) and (112) Lagrange's equations will become:

$$\frac{I_0}{4\beta_i}\ddot{q}_i + \frac{GI_p\beta_i}{4l}q_i = \frac{1}{A_i}Q_i$$

or

$$\ddot{q}_i + \frac{a^2\beta_i^2}{l^2}q_i = \frac{4\beta_i}{I_0A_i}Q_i, \qquad (l)$$

in which Q_i is the symbol for the generalized force corresponding to the generalized coordinate q_i.

Considering only the vibration produced by the disturbing force, we obtain from eq. (l)

$$q_i = \frac{4l}{aI_0A_i}\int_0^t Q_i \sin\frac{a\beta_i}{l}(t - t_1)dt_1.$$

Substituting in eq. (108)[1], the general expression for the vibrations produced by the disturbing forces, we will find:

$$\theta = \frac{4l}{aI_0}\sum_{i=1}^{i=\infty}\frac{1}{A_i}\left(\cos\frac{\beta_i x}{l} - m\beta_i \sin\frac{\beta_i x}{l}\right)\int_0^t Q_i \sin\frac{a\beta_i}{l}(t - t_1)dt_1. \qquad (113)$$

In every particular case it remains only to substitute for Q_i the corresponding expression and to perform the indicated integration in order to obtain forced vibrations. These forced vibrations have the tendency to increase indefinitely * when the period of the disturbing force coincides with the period of one of the natural vibrations.

40. Lateral Vibration of Prismatical Bars.—*Differential Equation of Lateral Vibration.*—Assuming that vibration occurs in one of the principal planes of flexure of the bar and that cross sectional dimensions are small in comparison with the length of the bar, the well known differential equation of the deflection curve

$$EI\frac{d^2y}{dx^2} = -M \qquad (114)$$

will now be used, in which

EI = flexural rigidity and,

M = bending moment at any cross section. The direction of the axes and the positive directions of bending moments and shearing forces are as shown in Fig. 115.

* Damping is neglected in our calculations.

Differentiating eq. (114) twice we obtain

$$\frac{d}{dx}\left(EI\frac{d^2y}{dx^2}\right) = -\frac{dM}{dx} = -Q,$$
$$\frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right) = -\frac{dQ}{dx} = q. \tag{a}$$

This last equation representing the differential equation of a bar subjected to a continuous load of intensity q can be used also for obtaining the equation of lateral vibration. It is only necessary to apply D'Alembert's principle and to imagine that the vibrating bar is loaded by inertia forces, the intensity of which varies along the length of the bar and is given by

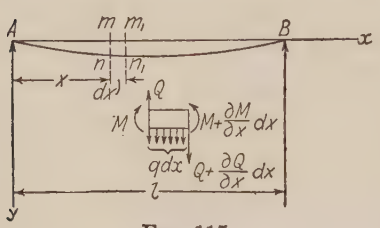


FIG. 115.

$$-\frac{\gamma A}{g}\frac{\partial^2 y}{\partial t^2}, \tag{b}$$

where γ = weight of material of the bar per unit volume,
A = cross sectional area.

Substituting (b) for q in eq. (a) the general equation for the lateral vibration of the bar becomes

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right) = -\frac{\gamma A}{g}\frac{\partial^2 y}{\partial t^2}. \tag{115}$$

In the particular case of a prismatical bar the flexural rigidity EI remains constant along the length of the bar and we obtain from eq. (115)

$$EI\frac{\partial^4 y}{\partial x^4} = -\frac{\gamma A}{g}\frac{\partial^2 y}{\partial t^2}$$

or

$$\frac{\partial^2 y}{\partial t^2} + a^2\frac{\partial^4 y}{\partial x^4} = 0, \tag{116}$$

in which

$$a^2 = \frac{EIg}{A\gamma}. \tag{117}$$

We begin with studying the *normal modes* of vibration. When a bar performs a normal mode of vibration the deflection at any location varies harmonically with the time and can be represented as follows:

$$y = X(A\cos pt + B\sin pt), \tag{c}$$

where X is a function of the coordinate x determining the shape of the normal mode of vibration under consideration. Such functions are called "normal functions." Substituting (c) in eq. (116), we obtain,

$$\frac{d^4X}{dx^4} = \frac{p^2}{a^2}X, \tag{118}$$

from which the normal functions for any particular case can be obtained.

By using the notation

$$\frac{p^2}{a^2} = \frac{p^2A\gamma}{EIg} = k^4 \tag{119}$$

it can be easily verified that $\sin kx$, $\cos kx$, $\sinh kx$ and $\cosh kx$ will be particular solutions of eq. (118) and the general solution of this equation will be obtained in the form,

$$X = C_1 \sin kx + C_2 \cos kx + C_3 \sinh kx + C_4 \cosh kx, \tag{120}$$

in which $C_1, \cdots C_4$ are arbitrary constants which should be determined in every particular case from the conditions at the ends of the bar. At an end which is simply supported, i.e., where the deflection and bending moment are equal to zero, we have

$$X = 0; \qquad \frac{d^2X}{dx^2} = 0. \tag{d}$$

At a built-in end, i.e., where the deflection and slope of the deflection curve are equal to zero,

$$X = 0; \qquad \frac{dX}{dx} = 0. \tag{e}$$

At a free end the bending moment and the shearing force are equal to zero and we obtain,

$$\frac{d^2X}{dx^2} = 0; \qquad \frac{d^3X}{dx^3} = 0. \tag{f}$$

For the two ends of a vibrating bar we always will have four end conditions from which the ratios between the arbitrary constants of the general solution (120) and the *frequency equation* can be obtained. In this manner the modes of natural vibration and their frequencies will be established. By superimposing all possible normal vibrations (c) the general expression for the free lateral vibrations becomes:

$$y = \sum_{i=1}^{i=\infty} X_i(A_i \cos p_it + B_i \sin p_it)\cdots. \tag{121}$$

Applications of this general theory to particular cases will be considered later.

Forced Vibration.—In considering forced lateral vibrations of bars *generalized coordinates* are very useful and, in the following, the expressions in the brackets of eq. (121) will be taken as such coordinates. Denoting them by the symbol q_i we obtain

$$y = \sum_{i=1}^{i=\infty} q_i X_i. \tag{122}$$

In order to derive Lagrange's equations it is necessary to find expressions for the potential and kinetic energy.

The potential energy of the system is the energy of bending and can be calculated as follows:

$$V = \frac{EI}{2}\int_0^l \left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx = \frac{EI}{2}\sum_{i=1}^{i=\infty} q_i^2 \int_0^l \left(\frac{d^2 X_i}{dx^2}\right)^2 dx. \tag{123}$$

The kinetic energy of the vibrating bar will be

$$T = \frac{\gamma A}{2g}\int_0^l \dot{y}^2 dx = \frac{\gamma A}{2g}\sum_{i=1}^{i=\infty} \dot{q}_i^2 \int_0^l X_i^2 dx. \tag{124}$$

The terms containing products of the coordinates disappear from the expressions (123) and (124) in virtue of the fundamental property of normal functions (see p. 127). This can also be proven by direct integration.

Let X_m and X_n be two normal functions corresponding to normal modes of vibration of the order m and n, having frequencies $p_m/2\pi$ and $p_n/2\pi$. Substituting in eq. (118) we obtain

$$\frac{d^4 X_m}{dx^4} = \frac{p_m^2}{a^2} X_m,$$

$$\frac{d^4 X_n}{dx^4} = \frac{p_n^2}{a^2} X_n.$$

Multiplying the first of these equations with X_n and the second with X_m, subtracting one from another and integrating we have

$$\frac{p_n^2 - p_m^2}{a^2}\int_0^l X_m X_n dx = \int_0^l \left(X_m \frac{d^4 X_n}{dx^4} - X_n \frac{d^4 X_m}{dx^4}\right) dx,$$

from which, by integration by parts, follows

$$\frac{p_n^2 - p_m^2}{a^2}\int_0^l X_m X_n dx = \left| X_m \frac{d^3X_n}{dx^3} - X_n \frac{d^3X_m}{dx^3} + \frac{dX_n}{dx}\frac{d^2X_m}{dx^2} - \frac{dX_m}{dx}\frac{d^2X_n}{dx^2} \right|_0^l \cdots \quad (125)$$

From the end conditions (d), (e) and (f) it can be concluded that in all cases the right side of the above equation is equal to zero, hence,

$$\int_0^l X_m X_n dx = 0 \text{ when } m \neq n$$

and the terms containing the products of the coordinates disappear from eq. (124). By using an analogous method it can be shown also that the products of the coordinates disappear from eq. (123).

Equation (125) can be used also for the calculation of integrals such as

$$\int_0^l X_m^2 dx \quad \text{and} \quad \int_0^l \left(\frac{d^2X_m}{dx^2}\right)^2 dx \qquad (g)$$

entering into the expressions (123) and (124) of the potential and the kinetic energy of a vibrating bar.

It is easy to see that by directly substituting $m = n$ into this equation, the necessary results cannot be obtained because both sides of the equation become equal to zero. Therefore the following procedure should be adopted for calculating the integrals (g). Substitute for X_n in eq. (125) a function which is very near to the function X_m and which will be obtained from equations (118) and (119) by giving to the quantity k an infinitely small increase δk, so that X_n approaches X_m when δk approaches zero. Then

$$\frac{p_n^2}{a^2} = (k + \delta k)^4 = k^4 + 4k^3\delta k,$$

$$\frac{p_n^2 - p_m^2}{a^2} = 4k^3\delta k,$$

$$X_n = X_m + \frac{dX_m}{dk}\delta k.$$

Substituting in eq. 125 and neglecting small quantities of higher order we obtain

$$4k^3\int_0^l X_m^2 dx = \left| X_m \frac{d}{dk}\frac{d^3X_m}{dx^3} - \frac{dX_m}{dk}\frac{d^3X_m}{dx^3} + \frac{d}{dk}\left(\frac{dX_m}{dx}\right)\frac{d^2X_m}{dx^2} - \frac{dX_m}{dx}\frac{d}{dk}\left(\frac{d^2X_m}{dx^2}\right) \right|_0^l. \quad (h)$$

In the following we denote by X', X'', $\cdots$ consecutive derivatives of X with respect to kx, then

$$\frac{dX_m}{dx} = kX_m'; \qquad \frac{dX_m}{dk} = xX_m'.$$

With these notations eq. (118) becomes

$$X'''' = X,$$

and eq. (h) will have the following form:

$$4k^3 \int_0^l X_m^2 dx = \Big| 3X_m k^2 X_m''' + k^3 x X_m^2 - k^3 x X_m' X_m''' + \\ + k^2 X_m''(X_m' + kxX_m'') - kX_m'(2kX_m'' + k^2 x X_m''') \Big|_0^l$$

or

$$4k \int_0^l X_m^2 dx = \Big| 3X_m X_m''' + kxX_m^2 - 2kxX_m' X_m''' - X_m' X_m'' + kx(X_m'')^2 \Big|_0^l . \quad (k)$$

From the end conditions (d), (e) and (f) it is easy to see that the terms in eq. (k) containing the products $X_m X_m'''$ and $X_m' X_m''$ are equal to zero for any manner of fastening the ends, hence

$$\int_0^l X_m^2 dx = \tfrac{1}{4} \Big| x\{X_m^2 - 2X_m' X_m''' + (X_m'')^2\} \Big|_0^l \\ = \frac{l}{4} \{X_m^2 - 2X_m' X_m''' + (X_m'')^2\}_{x=l}. \quad (126)$$

From this equation the first of the integrals (g) easily can be calculated for any kind of fastening of the ends of the bar. If the right end ($x = l$) of the bar is free,

$$(X_m'')_{x=l} = 0; \qquad (X_m''')_{x=l} = 0,$$

and we obtain, from (126)

$$\int_0^l X_m^2 dx = \frac{l}{4} (X_m^2)_{x=l}. \quad (127)$$

If the same end is built in, we obtain

$$\int_0^l X_m^2 dx = \frac{l}{4} (X_m'')_{x=l}^2. \quad (128)$$

For the hinged end we obtain

$$\int_0^l X_m^2 dx = -\frac{l}{2} (X_m' X_m''')_{x=l}. \quad (129)$$

In calculating the second of the integrals (g) equation (118) should be used. Multiplying this equation by X and integrating along the length of the bar:

$$\frac{p^2}{a^2} \int_0^l X^2 dx = \int_0^l \frac{d^4 X}{dx^4} X dx.$$

Integrating the right side of this equation by parts we obtain,

$$\int_0^l \left(\frac{d^2 X}{dx^2}\right)^2 dx = \frac{p^2}{a^2} \int_0^l X^2 dx. \quad (130)$$

This result together with eq. (126) gives us the second of the integrals (g) and now the expressions (123) and (124) for V and T can be calculated. Eqs. (126) and (130) are very useful in investigating forced vibrations of bars with other end conditions than hinged ones.

41. The Effect of Shearing Force and Rotatory Inertia.—In the previous discussion the cross sectional dimensions of the bar were considered to be very small in comparison with the length and the simple equation (114) was used for the deflection curve. Corrections will now be given, taking into account the effect of the cross sectional dimensions

on the frequency. These corrections may be of considerable importance in studying the modes of vibration of higher frequencies when a vibrating bar is subdivided by *nodal cross sections* into comparatively short portions.

*Rotatory Inertia.**—It is easy to see that during vibration the elements of the bar such as mnm_1n_1 (see Fig. 115) perform not only a translatory motion but also rotate. The variable angle of rotation which is equal to the slope of the deflection curve will be expressed by $\partial y/\partial x$ and the corresponding angular velocity and angular acceleration will be given by

$$\frac{\partial^2 y}{\partial x \partial t} \quad \text{and} \quad \frac{\partial^3 y}{\partial x \partial t^2}.$$

Therefore the moment of the inertia forces of the element mnm_1n_1 about the axis through its center of gravity and perpendicular to the xy plane will be †

$$-\frac{I\gamma}{g}\frac{\partial^3 y}{\partial x \partial t^2}dx.$$

This moment should be taken into account in considering the variation in bending moment along the axis of the bar. Then, instead of the first of the equations (*a*) p. 222 we will have,

$$\frac{dM}{dx} = Q - \frac{I\gamma}{g}\frac{\partial^3 y}{\partial x \partial t^2}. \tag{a}$$

Substituting this value of dM/dx in the equation for the deflection curve

$$EI\frac{d^4 y}{dx^4} = -\frac{d^2 M}{dx^2}$$

and using (*b*) p. 222, we obtain

$$EI\frac{\partial^4 y}{\partial x^4} = -\frac{\gamma A}{g}\frac{\partial^2 y}{\partial t^2} + \frac{I\gamma}{g}\frac{\partial^4 y}{\partial x^2 \partial t^2}. \tag{131}$$

This is the differential equation for the lateral vibration of prismatical bars in which the second term on the right side represents the effect of rotatory inertia.

Effect of Shearing Force.‡—A still more accurate differential equation is obtained if not only the rotatory inertia, but also the deflection due to shear will be taken into account. The slope of the deflection curve depends not only on the rotation of cross sections of the bar but also on the shear. Let ψ denote the slope of the deflection curve when the shearing force is neglected and β the angle of shear at the neutral axis in the same cross section, then we find for the complete slope

$$\frac{dy}{dx} = \psi + \beta.$$

From the elementary theory of bending we have for bending moment and shearing force the following equations,

$$M = -EI\frac{d\psi}{dx}; \qquad Q = k'\beta AG = k'\left(\frac{dy}{dx} - \psi\right)AG, \tag{b}$$

* See Lord Rayleigh, "Theory of Sound," paragraph 186.

† The moment is taken positive when it is a clockwise direction.

‡ See writer's paper in Philosophical Magazine (Ser. 6) Vol. 41, p. 744 and Vol. 43, p. 125.

in which k' is a numerical factor depending on the shape of the cross section; A is the cross sectional area and G = modulus of elasticity at shear. The differential equation of rotation of an element mnm_1n_1 (Fig. 115) will be

$$-\frac{\partial M}{\partial x}dx + Qdx = \frac{I\gamma}{g}\frac{\partial^2\psi}{\partial t^2}dx.$$

Substituting (b) we obtain

$$EI\frac{\partial^2\psi}{\partial x^2} + k'\left(\frac{\partial y}{\partial x} - \psi\right)AG - \frac{I\gamma}{g}\frac{\partial^2\psi}{\partial t^2} = 0. \qquad (c)$$

The differential equation for the translatory motion of the same element in a vertical direction will be

$$\frac{\partial Q}{\partial x}dx = \frac{\gamma A}{g}\frac{\partial^2 y}{\partial t^2}dx$$

or

$$\frac{\gamma A}{g}\frac{\partial^2 y}{\partial t^2} - k'\left(\frac{\partial^2 y}{\partial x^2} - \frac{\partial\psi}{\partial x}\right)AG = 0. \qquad (d)$$

Eliminating ψ from equations (c) and (d) the following more complete differential equation for the lateral vibration of prismatical bars will be obtained

$$EI\frac{\partial^4 y}{\partial x^4} + \frac{\gamma A}{g}\frac{\partial^2 y}{\partial t^2} - \left(\frac{\gamma I}{g} + \frac{EI\gamma}{gk'G}\right)\frac{\partial^4 y}{\partial x^2\partial t^2} + \frac{\gamma I}{g}\frac{\gamma}{gk'G}\frac{\partial^4 y}{\partial t^4} = 0. \qquad (132)$$

The application of this equation in calculating the frequencies will be shown in the following paragraph.

42. Free Vibration of a Bar with Hinged Ends.—*General Solution.*—In considering particular cases of vibration it is useful to present the general solution (120) in the following form

$$\begin{aligned} X = {} & C_1(\cos kx + \cosh kx) + C_2(\cos kx - \cosh kx) \\ & + C_3(\sin kx + \sinh kx) + C_4(\sin kx - \sinh kx)\cdots. \end{aligned} \qquad (133)$$

In the case of hinged ends the end conditions are

$$\begin{aligned} &(1)\quad (X)_{x=0} = 0; \qquad (2)\quad \left(\frac{d^2X}{dx^2}\right)_{x=0} = 0; \\ &(3)\quad (X)_{x=l} = 0; \qquad (4)\quad \left(\frac{d^2X}{dx^2}\right)_{x=l} = 0. \end{aligned} \qquad (a)$$

From the first two conditions (a) it can be concluded that the constants C_1 and C_2 in solution (133) should be taken equal to zero. From conditions (3) and (4) we obtain $C_3 = C_4$ and

$$\sin kl = 0, \qquad (134)$$

which is the *frequency equation* for the case under consideration. The consecutive roots of this equation are

$$kl = \pi, 2\pi, 3\pi, \cdots. \qquad (135)$$

The frequencies of the consecutive modes of vibration will be obtained from eq. (119):

$$p_1 = ak_1^2 = \frac{a\pi^2}{l^2}; \qquad p_2 = \frac{4a\pi^2}{l^2}; \qquad p_3 = \frac{9a\pi^2}{l^2}; \cdots, \tag{136}$$

and the frequency f_n of any mode of vibration will be found from the equation

$$f_n = \frac{p_n}{2\pi} = \frac{n^2 a\pi}{2l^2} = \frac{\pi n^2}{2l^2}\sqrt{\frac{EIg}{A\gamma}}. \tag{137}$$

The corresponding period of vibration will be

$$\tau_n = \frac{1}{f_n} = \frac{2l^2}{\pi n^2}\sqrt{\frac{A\gamma}{EIg}}. \tag{137^1}$$

It is seen that the period of vibration is proportional to the square of the length and inversely proportional to the radius of gyration of the cross section. For geometrically similar bars the periods of vibration increase in the same proportion as the linear dimensions.

In the case of rotating circular shafts of uniform cross section the frequencies calculated by eq. 137 represent the critical numbers of revolutions per second. When the speed of rotation of the shaft approaches one of the frequencies (137) a considerable lateral vibration of the shaft should be expected.

The shape of the deflection curve for the various modes of vibration is determined by the normal function (133). It was shown that in the case we are considering, $C_1 = C_2 = 0$ and $C_3 = C_4$, hence

$$X = D \sin kx. \tag{b}$$

Substituting for k its values, from eq. (135), we obtain

$$X_1 = D_1 \sin\frac{\pi x}{l}; \qquad X_2 = D_2 \sin\frac{2\pi x}{l}; \qquad X_3 = D_3 \sin\frac{3\pi x}{l}; \cdots.$$

It is seen that the deflection curve during vibration is a sine curve, the number of half waves in the consecutive modes of vibration being equal to 1, 2, 3 $\cdots$. By superimposing such sinusoidal vibrations any kind of free vibration due to any initial conditions can be represented. Substituting (b) in the general solution (121) we obtain

$$y = \sum_{i=1}^{i=\infty} \sin\frac{i\pi x}{l}(A_i \cos p_i t + B_i \sin p_i t). \tag{138}$$

The arbitrary constants A_i, B_i of this solution should be determined in every particular case so as to satisfy the initial conditions. Assume, for instance, that the initial deflections and initial velocities along the bar are given by the equations

$$(y)_{t=0} = f(x) \qquad \text{and} \qquad (\dot{y})_{t=0} = f_1(x).$$

Substituting $t = 0$ in expression (138) and in the derivative of this expression with respect to t, we obtain,

$$(y)_{t=0} = f(x) = \sum_{i=1}^{i=\infty} A_i \sin\frac{i\pi x}{l}, \tag{c}$$

$$(\dot{y})_{t=0} = f_1(x) = \sum_{i=1}^{i=\infty} p_i B_i \sin\frac{i\pi x}{l}. \tag{d}$$

Now the constants A_i and B_i can be calculated in the usual way by multiplying (c) and (d) by $\sin(i\pi x/l)dx$ and by integrating both sides of these equations from $x = 0$ to $x = l$. In this manner we obtain

$$A_i = \frac{2}{l}\int_0^l f(x)\sin\frac{i\pi x}{l}\,dx, \tag{e}$$

$$B_i = \frac{2}{lp_i}\int_0^l f_1(x)\sin\frac{i\pi x}{l}\,dx. \tag{f}$$

Assume, for instance, that in the initial moment the axis of the bar is straight and that due to impact an initial velocity v is given to a short portion δ of the bar at the distance c from the left support. Then, $f(x) = 0$ and $f_1(x)$ also is equal to zero in all points except the point $x = c$ for which $f_1(c) = v$. Substituting this in the eqs. (e) and (f) we obtain,

$$A_i = 0; \qquad B_i = \frac{2}{lp_i} v\delta \sin\frac{i\pi c}{l}.$$

Substituting in (138)

$$y = \frac{2v\delta}{l}\sum_{i=1}^{i=\infty}\frac{1}{p_i}\sin\frac{i\pi c}{l}\sin\frac{i\pi x}{l}\sin p_i t. \tag{139}$$

If $c = (l/2)$, i.e., the impact is produced at the middle of the span,

$$\begin{aligned} y &= \frac{2v\delta}{l}\left(\frac{1}{p_1}\sin\frac{\pi x}{l}\sin p_1 t - \frac{1}{p_3}\sin\frac{3\pi x}{l}\sin p_3 t + \frac{1}{p_5}\sin\frac{5\pi x}{l}\sin p_5 t - \cdots\right) \\ &= \frac{2v\delta l}{a\pi^2}\left(\frac{1}{1}\sin\frac{\pi x}{l}\sin p_1 t - \frac{1}{9}\sin\frac{3\pi x}{l}\sin p_3 t + \frac{1}{25}\sin\frac{5\pi x}{l}\sin p_5 t - \cdots\right). \end{aligned} \tag{g}$$

It is seen that in this case only modes of vibration symmetrical about the middle of the span will be produced and the amplitudes of consecutive modes of vibration entering in eq. (g) decrease as $1/i^2$.

The Effect of Rotatory Inertia and of Shear.—In order to find the values of the frequencies more accurately equation (132) instead of equation (116) should be taken. Dividing eq. (132) by $A\gamma/g$ and using the notation,

$$r^2 = \frac{I}{A} \tag{h}$$

we obtain

$$a^2 \frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial t^2} - r^2 \left(1 + \frac{E}{k'G}\right) \frac{\partial^4 y}{\partial x^2 \partial t^2} + r^2 \frac{\gamma}{gk'G} \frac{\partial^4 y}{\partial t^4} = 0. \tag{132^1}$$

This equation and the end conditions will be satisfied by taking

$$y = C \sin \frac{m\pi x}{l} \cos p_m t. \tag{k}$$

Substituting in (132)^1 we obtain the following equation for calculating the frequencies

$$a^2 \frac{m^4\pi^4}{l^4} - p_m^2 - p_m^2 \frac{m^2\pi^2 r^2}{l^2} - p_m^2 \frac{m^2\pi^2 r^2}{l^2} \frac{E}{k'G} + \frac{r^2\gamma}{gk'G} p_m^4 = 0. \tag{140}$$

Considering only the first two terms in this equation we have

$$p_m = a \frac{m^2\pi^2}{l^2} = \frac{a\pi^2}{\lambda^2}, \tag{l}$$

in which

$\lambda = (l/m)$ = the length of the half waves in which the bar is subdivided during vibration.

This coincides with the result (136) obtained before. By taking the three first terms in eq. (140) and considering $\pi^2 r^2/\lambda^2$ as a small quantity we obtain

$$p_m = \frac{a\pi^2}{\lambda^2} \left(1 - \frac{\pi^2 r^2}{2\lambda^2}\right). \tag{m}$$

In this manner the *effect of rotatory inertia* is taken into account and we see that this correction becomes more and more important with a decrease of λ, i.e. with an increase in the frequency of vibration.

In order to obtain the *effect of shear* all terms of eq. (140) should be taken into consideration. Substituting the first approximation (l) for p_m in the last term of this equation it can be shown that this term is a small quantity of the second order as compared with the small quantity $\pi^2 r^2/\lambda^2$. Neglecting this term we obtain,

$$p_m = \frac{a\pi^2}{\lambda^2} \left\{1 - \frac{1}{2} \frac{\pi^2 r^2}{\lambda^2} \left(1 + \frac{E}{k'G}\right)\right\}. \tag{141}$$

Assuming $E = 8/3G$ and taking a bar of rectangular cross section for which $k' = 2/3$, we have

$$\frac{E}{k'G} = 4.$$

The correction due to shear is four times larger than the correction due to rotatory inertia.

Assuming that the wave length λ is ten times larger than the depth of the beam, we obtain

$$\frac{1}{2} \cdot \frac{\pi^2 r^2}{\lambda^2} = \frac{1}{2} \cdot \frac{\pi^2}{12} \cdot \frac{1}{100} = .004$$

and the correction for rotatory inertia and shear together will be about 2%.

43. Other End Fastenings.—*Bar with Free Ends.*—In this case we have the following end conditions:

$$\begin{aligned} &(1) \quad \left(\frac{d^2X}{dx^2}\right)_{x=0} = 0; \qquad (2) \quad \left(\frac{d^3X}{dx^3}\right)_{x=0} = 0; \\ &(3) \quad \left(\frac{d^2X}{dx^2}\right)_{x=l} = 0; \qquad (4) \quad \left(\frac{d^3X}{dx^3}\right)_{x=l} = 0. \end{aligned} \tag{a}$$

In order to satisfy the conditions (1) and (2) we have to take in the general solution (133)

$$C_2 = C_4 = 0$$

so that

$$X = C_1(\cos kx + \cosh kx) + C_3(\sin kx + \sinh kx). \tag{b}$$

From the conditions (3) and (4) we obtain

$$\begin{aligned} C_1(-\cos kl + \cosh kl) + C_3(-\sin kl + \sinh kl) &= 0, \\ C_1(\sin kl + \sinh kl) + C_3(-\cos kl + \cosh kl) &= 0. \end{aligned} \tag{c}$$

A solution for the constants C_1 and C_3, different from zero, can be obtained only in the case when the determinant of equations (*c*) is equal to zero. In this manner the following *frequency equation* is obtained:

$$(-\cos kl + \cosh kl)^2 - (\sinh^2 kl - \sin^2 kl) = 0$$

or, remembering that

$$\cosh^2 kl - \sinh^2 kl = 1,$$
$$\cos^2 kl + \sin^2 kl = 1,$$

we have

$$\cos kl \cosh kl = 1. \tag{142}$$

The first six consecutive roots of this equation are given in the table below:

k_1l	k_2l	k_3l	k_4l	k_5l	k_6l
0	4.730	7.853	10.996	14.137	17.279

Now the frequencies can be calculated by using equation (119)

$$f_1 = 0; \qquad f_2 = \frac{p_2}{2\pi} = \frac{k_2^2 a}{2\pi}; \qquad f_3 = \frac{p_3}{2\pi} = \frac{k_3^2 a}{2\pi}, \cdots.$$

Substituting the consecutive roots of eq. (142) in eq. (*c*) the ratios C_1/C_3 for the corresponding modes of vibration can be calculated and the shape of the deflection curve during vibration will then be obtained from eq. (*b*). In the figure 116 below the first three modes of natural vibration are shown. On these vibrations a displacement of the bar as a rigid body can be superposed. This displacement corresponds to the frequency $k_1 l = 0$. Then the right side of eq. (118) becomes zero and by taking into consideration the end conditions (*a*), we obtain $X = a + bx$. The corresponding motion can be investigated in the same manner as was shown in the case of longitudinal vibration (see p. 207).

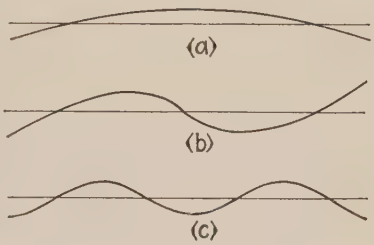


FIG. 116.

Bar with Built-in Ends.—The end conditions in this case are:

$$\begin{aligned} &(1)\quad (X)_{x=0} = 0; \qquad (2)\quad \left(\frac{dX}{dx}\right)_{x=0} = 0; \\ &(3)\quad (X)_{x=l} = 0; \qquad (4)\quad \left(\frac{dX}{dx}\right)_{x=l} = 0. \end{aligned} \tag{d}$$

The first two conditions will be satisfied if in the general solution (133) we take

$$C_1 = C_3 = 0.$$

From the two other conditions the following equations will be obtained

$$\begin{aligned} C_2(\cos kl - \cosh kl) + C_4(\sin kl - \sinh kl) &= 0, \\ C_2(\sin kl + \sinh kl) + C_4(-\cos kl + \cosh kl) &= 0, \end{aligned}$$

from which the same frequency equation as above (see eq. 142) can be deduced. This means that the consecutive frequencies of vibration of a bar with built-in ends are the same as for the same bar with free ends.*

Bar with One End Built-in, Other End Free.—Assuming that the left end ($x = 0$) is built in, the following end conditions will be obtained:

$$\begin{aligned} &(1)\quad (X)_{x=0} = 0; \qquad (2)\quad \left(\frac{dX}{dx}\right)_{x=0} = 0; \\ &(3)\quad \left(\frac{d^2X}{dx^2}\right)_{x=l} = 0; \qquad (4)\quad \left(\frac{d^3X}{dx^3}\right)_{x=l} = 0. \end{aligned}$$

* From eq. (118), it can be concluded that in this case there is no motion corresponding to $k_1 l = 0$.

From the first two conditions we conclude that $C_1 = C_3 = 0$ in the general solution (133). The remaining two conditions give us the following frequency equation:

$$\cos kl \cosh kl = -1.$$

The consecutive roots of this equation are given in the table below:

k_1l	k_2l	k_3l	k_4l	k_5l	k_6l
1.875	4.694	7.855	10.996	14.137	17.279

It is seen that with increasing frequency these roots approach the roots obtained above for a bar with free ends. The frequency of vibration of any mode will be

$$f_i = \frac{p_i}{2\pi} = \frac{ak_i^2}{2\pi}.$$

Taking, for instance, the fundamental mode of vibration, we obtain

$$f_1 = \frac{a}{2\pi}\left(\frac{1.875}{l}\right)^2.$$

The corresponding period of vibration will be:

$$\tau_1 = \frac{1}{f_1} = \frac{2\pi}{a}\frac{l^2}{1.875^2} = \frac{2\pi}{3.515}\sqrt{\frac{A\gamma l^4}{EIg}}.$$

This differs by less than 1.5 per cent. from the approximate solution obtained by using Rayleigh's method (see p. 58).

Bar with One End Built in, Other End Supported.—In this case the frequency equation will be

$$\tan kl = \tanh kl.$$

The consecutive roots of this equation are:

k_1l	k_2l	k_3l	k_4l	k_5l
3.927	7.069	10.210	13.352	16.493

*Beam on Many Supports.**—Consider the case of a continuous beam with n spans simply supported at the ends and at $(n - 1)$ intermediate supports. Let $l_1, l_2, \cdots, l_n$ the lengths of consecutive spans, the flexural rigidity of the beam being the same for all spans. Taking the origin of coordinates at the left end of each span, solution (120) p. 223

* See E. R. Darnley, Phil. Mag., Vol. 41 (1921), p. 81. See also D. M. Smith, Engineering, Vol. 120 (1925), p. 808.

will be used for the shape of the deflection curve of each span during vibration. Taking into consideration that the deflection at the left end ($x = 0$) is equal to zero the normal function for the span r will be

$$X_r = a_r(\cos kx - \cosh kx) + c_r \sin kx + d_r \sinh kx, \qquad (e)$$

in which a_r, c_r and d_r are arbitrary constants. The consecutive derivatives of (e) will be

$$X_r' = -a_r k(\sin kx + \sinh kx) + c_r k \cos kx + d_r k \cosh kx, \qquad (f)$$

$$X_r'' = -a_r k^2(\cos kx + \cosh kx) - c_r k^2 \sin kx + d_r k^2 \sinh kx. \qquad (g)$$

Substituting $x = 0$ in eqs. (f) and (g) we obtain

$$(X_r')_{x=0} = k(c_r + d_r); \qquad (X_r'')_{x=0} = -2k^2 a_r.$$

It is seen that $c_r + d_r$ is proportional to the slope of the deflection curve, and a_r is proportional to the bending moment at the support r. From the conditions of simply supported ends it can now be concluded that $a_1 = a_{n+1} = 0$.

Considering the conditions at the right end of the span r we have,

$$(X_r)_{x=l_r} = 0; \quad (X_r')_{x=l_r} = (X'_{r+1})_{x=0}; \quad (X_r'')_{x=l_r} = (X''_{r+1})_{x=0},$$

or by using (e), (f) and (g),

$$a_r(\cos kl_r - \cosh kl_r) + c_r \sin kl_r + d_r \sinh kl_r = 0, \qquad (h)$$

$$-a_r(\sin kl_r + \sinh kl_r) + c_r \cos kl_r + d_r \cosh kl_r = c_{r+1} + d_{r+1}, \qquad (k)$$

$$a_r(\cos kl_r + \cosh kl_r) + c_r \sin kl_r - d_r \sinh kl_r = 2a_{r+1}. \qquad (l)$$

Adding and subtracting (h) and (l) we obtain

$$a_r \cos kl_r + c_r \sin kl_r = a_r; \cosh kl_r - d_r \sinh kl_r = a_{r+1}$$

from which, provided $\sin kl_r$ is not zero,

$$c_r = \frac{a_{r+1} - a_r \cos kl_r}{\sin kl_r}; \qquad d_r = \frac{-a_{r+1} + a_r \cosh kl_r}{\sinh kl_r} \qquad (m)$$

and

$$c_r + d_r = a_r(\coth kl_r - \cot kl_r) - a_{r+1}(\operatorname{cosech} kl_r - \operatorname{cosec} kl_r). \qquad (n)$$

Using the notations:

$$\coth kl_r - \cot kl_r = \varphi_r, \qquad (o)$$
$$\operatorname{cosech} kl_r - \operatorname{cosec} kl_r = \psi_r,$$

we obtain

$$c_r + d_r = a_r\varphi_r - a_{r+1}\psi_r.$$

In the same manner for the span $r + 1$

$$c_{r+1} + d_{r+1} = a_{r+1}\varphi_{r+1} - a_{r+2}\psi_{r+1}. \tag{p}$$

Substituting (m) and (p) in eq. (k), we obtain

$$a_r\psi_r - a_{r+1}(\varphi_r + \varphi_{r+1}) + a_{r+2}\psi_{r+1} = 0. \tag{q}$$

Writing an analogous equation for each intermediate support the following system of $(n - 1)$ equations will be obtained:

$$\begin{aligned} -a_2(\varphi_1 + \varphi_2) + a_3\psi_2 &= 0, \\ a_2\psi_2 - a_3(\varphi_2 + \varphi_3) + a_4\psi_3 &= 0, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ a_{n-1}\psi_{n-1} - a_n(\varphi_{n-1} + \varphi_n) &= 0. \end{aligned} \tag{r}$$

Proceeding in the usual manner and putting equal to zero the determinant of these equations the frequency equation for the vibration of continuous beams will be obtained.

Take, for instance, a bar on three supports, then only one equation of the system (r) remains and the *frequency equation* will be

$$\varphi_1 + \varphi_2 = 0$$

or

$$\varphi_1 = -\varphi_2. \tag{s}$$

The frequencies of the consecutive modes of vibration will be obtained from the condition,

$$\varphi(kl_1) = -\varphi(kl_2).$$

In the solution of this transcendental equation it is convenient to draw a graph of the functions φ and $-\varphi$. In Fig. 117 φ and $-\varphi$ are given as functions of the argument kl expressed in degrees. The problem then reduces to finding by trial and error a line parallel to the x axis which cuts the graphs of φ and $-\varphi$ in points whose abscissae are in the ratio of the lengths of the spans.

Taking, for instance, $l_1 : l_2 = 6 : 4.5$, we obtain for the smallest root

$$kl_1 = 3.416,$$

from which the frequency of the fundamental mode of vibration becomes

$$f_1 = \frac{k_1^2 a}{2\pi} = \frac{3.416^2}{2\pi l_1^2}\sqrt{\frac{EIg}{A\gamma}}.$$

For the next higher frequency we obtain

$$kl_1 = 4.787.$$

The third frequency is given approximately by $kl_1 = 6.690$ so that the consecutive frequencies are in the ratio 1 : 1.96 : 3.82. If the lengths of the spans tend to become equal it is seen from Fig. 117 that the smallest root tends to $kl_1 = kl_2 = \pi$. In the case of the fundamental type of vibration each span will be in the condition of a bar with hinged ends. Another type of vibration will be obtained by assuming the tangent at the intermediate support to be horizontal, then each span will be in the condition of a bar with one end built in and another simply supported.

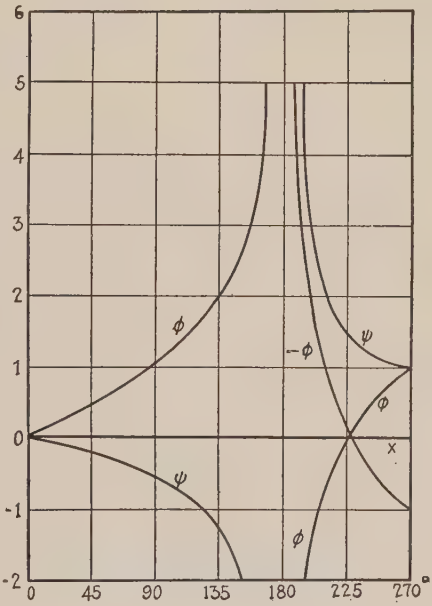


FIG. 117.

In the case of three spans we obtain, from (r),

$$-a_2(\varphi_1 + \varphi_2) + a_3\psi_2 = 0,$$
$$a_2\psi_2 - a_3(\varphi_2 + \varphi_3) = 0,$$

and the frequency equation becomes

$$(\varphi_1 + \varphi_2)(\varphi_2 + \varphi_3) - \psi_2^2 = 0. \qquad (t)$$

Having tables of the functions φ and ψ * the frequency of the fundamental mode can be found, from (t), by a process of trial and error.

44. Forced Vibration of a Beam with Supported Ends.—*General.*—In the case of a beam with supported ends the general expression for flexural

* Such tables are given in the paper by E. R. Darnley; loc. cit. p. 234. Another method by using nomographic solution is given in the paper by D. M. Smith, loc. cit., p. 234 in which the application of this problem to the vibration of condenser tubes is shown.

vibration is given by eq. (138). By using the symbols q_i for the generalized coordinates we obtain from the above equation

$$y = \sum_{i=1}^{i=\infty} q_i \sin \frac{i\pi x}{l}. \tag{a}$$

The expressions for the potential and kinetic energy will now be found from eqs. (123) and (124) by substituting $\sin i\pi x/l$ for X_i:

$$V = \frac{EI}{2} \sum_{i=1}^{i=\infty} q_i^2 \int_0^l \frac{i^4\pi^4}{l^4} \sin^2 \frac{i\pi x}{l}\, dx = \frac{EI\pi^4}{4l^3} \sum_{i=1}^{i=\infty} i^4 q_i^2, \tag{143}$$

$$T = \frac{A\gamma}{2g} \sum_{i=1}^{i=\infty} \dot{q}_i^2 \int_0^l \sin^2 \frac{i\pi x}{l}\, dx = \frac{A\gamma l}{4g} \sum_{i=1}^{i=\infty} \dot{q}_i^2. \tag{144}$$

If disturbing forces are acting on the beam, Lagrange's equation (72) for any coordinate q_i will be

$$\frac{A\gamma l}{2g} \ddot{q}_i + \frac{EI\pi^4 i^4}{2l^3} q_i = Q_i$$

or

$$\ddot{q}_i + \frac{i^4\pi^4 a^2}{l^4} q_i = \frac{2g}{A\gamma l} Q_i, \tag{b}$$

in which Q_i denotes the generalized force corresponding to the coordinate q_i and a^2 is given by eq. (117). The general solution of eq. (b) is

$$q_i = A_i \cos \frac{i^2\pi^2 at}{l^2} + B_i \sin \frac{i^2\pi^2 at}{l^2} + \\ + \frac{l^2}{i^2\pi^2 a} \frac{2g}{A\gamma l} \int_0^t Q_i \sin \frac{i^2\pi^2 a(t - t_1)}{l^2}\, dt_1. \tag{c}$$

The first two terms in this solution represent the free vibration determined by the initial conditions while the third term represents the vibration produced by the disturbing forces.

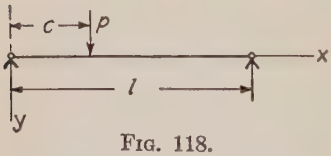


Fig. 118.

Pulsating Force.—As an example let us consider now the case of a pulsating force $P = P_0 \sin nt_1$ applied at a distance c from the left support (see Fig. 118). In order to obtain a generalized force Q_i assume that a small increase δq_i is given to a coordinate q_i. The corresponding deflection of the beam, from eq. (a), will be

$$\delta y = \delta q_i \sin \frac{i\pi x}{l}$$

and the work done by the external force P on this displacement is

$$P\delta q_i \sin\frac{i\pi c}{l}\cdot$$

Then,

$$Q_i = P\sin\frac{i\pi c}{l} = P_0 \sin\frac{i\pi c}{l}\sin nt_1. \qquad (d)$$

Substituting in eq. (c) and considering only that part of the vibrations produced by the pulsating force we obtain

$$q_i = \frac{2g}{A\gamma}P_0\sin\frac{i\pi c}{l}\left(\frac{l^3}{i^4\pi^4a^2 - n^2l^4}\sin nt - \right.$$
$$\left. - \frac{nl^5}{i^2\pi^2a(i^4\pi^4a^2 - n^2l^4)}\sin\frac{i^2\pi^2at}{l^2}\right). \qquad (e)$$

Substituting in eq. (a), we have

$$y = \frac{2gP_0l^3}{A\gamma}\sum_{i=1}^{i=\infty}\frac{\sin\frac{i\pi c}{l}\sin\frac{i\pi x}{l}}{i^4\pi^4a^2 - n^2l^4}\sin nt$$
$$- \frac{2gnP_0l^5}{A\gamma\pi^2a}\sum_{i=1}^{i=\infty}\frac{\sin\frac{i\pi c}{l}\sin\frac{i\pi x}{l}}{i^2(i^4\pi^4a^2 - n^2l^4)}\sin\frac{i^2\pi^2at}{l^2}\cdot \qquad (145)$$

It is seen that the first series in this solution is proportional to $\sin nt$. It has the same period as the disturbing force and represents *forced vibrations* of the beam. The second series represents *free vibrations* of the beam produced by application of the force. These latter vibrations due to various kinds of resistance will be gradually damped out and only the forced vibrations, given by equation

$$y = \frac{2gP_0l^3}{A\gamma}\sum_{i=1}^{i=\infty}\frac{\sin\frac{i\pi c}{l}\sin\frac{i\pi x}{l}}{i^4\pi^4a^2 - n^2l^4}\sin nt, \qquad (f)$$

are of practical importance.

If the pulsating force P is varying very slowly, n is a very small quantity and n^2l^4 can be neglected in the denominator of the series (f), then

$$y = \frac{2gPl^3}{A\gamma\pi^4a^2}\sum_{i=1}^{i=\infty}\frac{1}{i^4}\sin\frac{i\pi c}{l}\sin\frac{i\pi x}{l} \qquad (j)$$

or, by using eq. 117,

$$y = \frac{2Pl^3}{EI\pi^4}\sum_{i=1}^{i=\infty}\frac{1}{i^4}\sin\frac{i\pi c}{l}\sin\frac{i\pi x}{l}\cdot \qquad (g)$$

This expression represents the statical deflection of the beam produced by the load P.* In the particular case, when the force P is applied at the middle, $c = l/2$ and we obtain

$$y = \frac{2Pl^3}{EI\pi^4}\left(\sin\frac{\pi x}{l} - \frac{1}{3^4}\sin\frac{3\pi x}{l} + \frac{1}{5^4}\sin\frac{5\pi x}{l} - \cdots\right). \tag{h}$$

The series converges rapidly and a satisfactory approximation for the deflections will be obtained by taking the first term only. In this manner we find for the deflection at the middle:

$$(y)_{x=\frac{l}{2}} = \frac{2Pl^3}{EI\pi^4} = \frac{Pl^3}{48.7EI}.$$

The error of this approximation is about 1.5 per cent.

Denoting by α the ratio of the frequency of the disturbing force to the frequency of the fundamental type of free vibration, we obtain

$$\alpha = \frac{nl^2}{a\pi^2}$$

and the series (f), representing forced vibrations, becomes

$$y = \frac{2P_0 \sin ntl^3}{EI\pi^4}\sum_{i=1}^{i=\infty}\frac{\sin\frac{i\pi c}{l}\sin\frac{i\pi x}{l}}{i^4 - \alpha^2}.$$

If the pulsating force is applied at the middle, we obtain

$$y = \frac{2P_0 \sin ntl^3}{EI\pi^4}\left(\frac{\sin\frac{\pi x}{l}}{1-\alpha^2} - \frac{\sin\frac{3\pi x}{l}}{3^4-\alpha^2} + \frac{\sin\frac{5\pi x}{l}}{5^4-\alpha^2}\cdots\right). \tag{k}$$

For small α the first term of this series represents the deflection with good accuracy and comparing (k) with (h) it can be concluded that the ratio of the dynamical deflection to the statical deflection is approximately equal to

$$\frac{y_d}{y_s} = \frac{1}{1-\alpha^2}. \tag{l}$$

If, for instance, the frequency of the disturbing force is four times as small as the frequency of the fundamental mode of vibration, the dynamical deflection will be about 6 per cent greater than the statical deflection.

* See, "Applied Elasticity," p. 131.

Due to the fact that the problems on vibration of bars are represented by *linear* differential equations, the *principle of superposition* holds and if there are several pulsating forces acting on the beam, the resulting vibration will be obtained by superimposing the vibrations produced by the individual forces. The case of continuously distributed pulsating forces also can be solved in the same manner; the summation only has to be replaced by an integration along the length of the beam. Assume, for instance, that the beam is loaded by a uniformly distributed load of the intensity:

$$q = q_0 \sin nt.$$

Such a load condition exists, for instance, in a locomotive side rod under the action of lateral inertia forces. In order to determine the vibrations $q_0\, dc$ should be substituted for P_0 in eq. (f) and afterwards this equation should be integrated with respect to c within the limits $c = 0$ and $c = l$. In this manner we obtain

$$y = \frac{4gq_0l^4}{A\gamma\pi} \sum_{i=1}^{i=\infty} \frac{\sin \dfrac{i\pi x}{l}}{i(i^4\pi^4a^2 - n^2l^4)} \sin nt. \qquad (m)$$

If the frequency of the load is very small in comparison with the frequency of the fundamental mode of vibration of the bar the term n^2l^4 in the denominators of the series (m) can be neglected and we obtain,

$$y = \frac{4ql^4}{EI\pi^5}\left(\frac{\sin \dfrac{\pi x}{l}}{1^5} + \frac{\sin \dfrac{3\pi x}{l}}{3^5} + \frac{\sin \dfrac{5\pi x}{l}}{5^5} + \cdots\right). \qquad (n)$$

This very rapidly converging series represents the statical deflection of the beam produced by a uniformly distributed load q. By taking $x = l/2$ we obtain for the deflection at the middle

$$(y)_{x=\frac{l}{2}} = \frac{4ql^4}{EI\pi^5}\left(1 - \frac{1}{3^5} + \frac{1}{5^5} - \cdots\right). \qquad (p)$$

If only the first term of this series be taken, the error in the deflection at the middle will be about 1/4 per cent. If the frequency of the pulsating load is not small enough to warrant application of the statical equation, the same method can be used as was shown in the case of a single force and we will arrive at the same conclusion as represented by eq. (l).

Moving Constant Force.—If a constant vertical force P is moving along the length of a beam it produces vibrations which can be calculated without any difficulty by using the general equation (c). Let v denote

the constant velocity of the moving force and let the force be at the left support at the initial moment ($t = 0$), then at any other moment $t = t_1$ the distance of this force from this left support will be vt_1. In order to determine the generalized force Q_i for this case assume that the coordinate q_i in the general expression (a) of the deflection curve obtains an infinitely small increase δq_i. The work done by the force P due to this displacement will be

$$P(\delta y)_{x=vt_1} = P\delta q_i \sin\frac{i\pi vt_1}{l}.$$

Hence the generalized force

$$Q_i = P \sin\frac{i\pi vt_1}{l}.$$

Substituting this in the third term of equation (c) the following expression will be found for the vibrations produced by the moving load.*

$$y = \frac{2gPl^3}{A\gamma\pi^2}\sum_{i=1}^{i=\infty}\frac{\sin\frac{i\pi x}{l}}{i^2(i^2\pi^2a^2 - v^2l^2)}\sin\frac{i\pi vt}{l}$$

$$-\frac{2gPl^4v}{A\gamma\pi^3a}\sum_{i=1}^{i=\infty}\frac{\sin\frac{i\pi x}{l}}{i^3(i^2\pi^2a^2 - v^2l^2)}\sin\frac{i^2\pi^2at}{l^2}. \tag{146}$$

The first series in this solution represents forced vibrations and the second series free vibrations of the beam.

If the velocity v of the moving force be very small, we can put $v = 0$ and $vt = c$ in the solution above; then

$$y = \frac{2gPl^3}{A\gamma\pi^4a^2}\sum_{i=1}^{i=\infty}\frac{1}{i^4}\sin\frac{i\pi c}{l}\sin\frac{i\pi x}{l}.$$

This is the statical deflection of the beam produced by the load P applied at the distance c from the left support (see eq. (j)). By using the notation,

$$\alpha^2 = \frac{v^2l^2}{a^2\pi^2}, \tag{q}$$

* This problem is of practical interest in connection with the study of bridge vibrations. The first solution of this problem was given by A. N. Kriloff, Member of the Academy of Sciences in S. Petersburg; see, Mathematische Annalen, Vol. 61 (1905). See also writer's paper in the "Bulletin of the Polytechnical Institute in Kiew" (1908). (German translation in Zeitschr. f. Math. u. Phys., Vol. 59 (1911)). Prof. C. E. Inglis in the Proc. of The Inst. of Civil Engineers, Vol. 218 (1924), London, came to the same results.

the forced vibrations in the general solution (146) can be presented in the following form

$$y = \frac{2Pl^3}{EI\pi^4} \sum_{i=1}^{i=\infty} \frac{\sin \frac{i\pi x}{l} \sin \frac{i\pi vt}{l}}{i^2(i^2 - \alpha^2)}. \tag{r}$$

It is interesting to note that this deflection completely coincides with the statical deflection of a beam * on which in addition to the lateral load P applied at a distance $c = vt$ from the left support a longitudinal compressive force S is acting, such that

$$\frac{S}{S_{cr}} = \frac{Sl^2}{EI\pi^2} = \alpha^2. \tag{s}$$

Here S_{cr} denotes the known *critical* or *column load* for the beam.

From the eqs. (s) and (q) we obtain

$$\frac{Sl^2}{EI\pi^2} = \frac{v^2l^2}{a^2\pi^2}$$

or

$$S = \frac{v^2A\gamma}{g}. \tag{t}$$

The effect of this force on the statical deflection of the beam loaded by P is equivalent to the effect of the velocity of a moving force P on the deflection (r) representing forced vibrations.

By increasing the velocity v, a condition can be reached where one of the denominators in the series (146) becomes equal to zero and resonance takes place. Assume, for instance, that

$$a^2\pi^2 = v^2l^2. \tag{u}$$

In this case the period of the fundamental vibration of the beam, equal to $2l^2/a\pi$, becomes equal to $2l/v$ and is twice as great as the time required for the force P to pass over the beam. The denominators in the first terms of both series in eq. (146) become, under the condition (u) equal to zero and the sum of these two terms will be

$$\frac{2gPl^3}{A\gamma\pi^2} \sin \frac{\pi x}{l} \frac{\sin \frac{\pi vt}{l} - \frac{lv}{\pi a} \sin \frac{\pi^2 at}{l^2}}{\pi^2a^2 - v^2l^2}.$$

* See "Applied Elasticity," p. 163. By using the known expression for the statical deflection curve the finite form of the function, from which the series (r) has its origin, can be obtained.

This has the form 0/0 and can be presented in the usual way as follows (see p. 13)

$$-\frac{Pg}{\gamma A\pi v}t\cos\frac{\pi vt}{l}\sin\frac{\pi x}{l}+\frac{Pgl}{\gamma A\pi^2v^2}\sin\frac{\pi vt}{l}\sin\frac{\pi x}{l}. \qquad (v)$$

This expression has its maximum value when

$$t=\frac{l}{v}$$

and is then equal to

$$\frac{Pgl}{\gamma A\pi^2v^2}\left(\sin\frac{\pi vt}{l}-\frac{\pi vt}{l}\cos\frac{\pi vt}{l}\right)_{t=l/v}\sin\frac{\pi x}{l}=\frac{Pl^3}{EI\pi^3}\sin\frac{\pi x}{l}. \qquad (w)$$

Taking into consideration that the expression (v) represents a satisfactory approximation for the dynamical deflection given by equation (146) it can be concluded that the maximum dynamical deflection at the resonance condition (u) is about 50 per cent larger than the maximum statical deflection which is equal to

$$\frac{Pl^3}{48EI}.$$

It is interesting to note that the maximum dynamical deflection occurs when the force P is leaving the beam. At this moment the deflection under the force P is equal to zero, hence the work done by this force during the passing of the beam is also equal to zero. In order to explain the source of the energy accumulated in the vibrating beam during the passing over of the force P we should assume that there is no friction and the beam produces a reaction R in the direction of the normal (Fig. 119). In this case, from the condition of equilibrium it follows that there should exist a horizontal force, equal to $P(dy/dx)$. The work done by this force during its passage along the beam will be

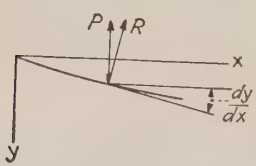


Fig. 119.

$$W=-\int_0^{l/v}P\left(\frac{dy}{dx}\right)_{x=vt}vdt.$$

Substituting expression (v) for (y) we obtain

$$W=-\frac{P^2g}{\gamma A\pi v^2}\int_0^{l/v}\left(\sin\frac{\pi vt}{l}-\frac{\pi vt}{l}\cos\frac{\pi vt}{l}\right)\cos\frac{\pi vt}{l}vdt=\frac{P^2gl}{\gamma A\pi^2v^2}\cdot\frac{\pi^2}{4}$$

or, by taking into consideration eqs. (u) and (117) we obtain

$$W = \frac{\pi^2}{4}\frac{P^2l^3}{EI\pi^4}.$$

This amount of work is very close * to the amount of the potential energy of bending in the beam at the moment $t = l/v$.

In the case of bridges, the time it takes to cross the bridge is usually large in comparison with the period of the fundamental type of vibration and the quantity α^2, given by eq. (q), is small. Then by taking only the first term in each series of eq. (146) and assuming that in the most unfavorable case the amplitudes of the forced and free vibrations are added to one another, we obtain for the maximum deflection,

$$\begin{aligned} y_{\max} &= \frac{2gPl^3}{\gamma A\pi^2}\left(\frac{1}{\pi^2a^2 - v^2l^2} + \frac{vl}{a\pi}\frac{1}{\pi^2a^2 - v^2l^2}\right) \\ &= \frac{2Pl^3}{EI\pi^4}\frac{1+\alpha}{1-\alpha^2} = \frac{2Pl^3}{EI\pi^4}\frac{1}{1-\alpha}. \end{aligned} \tag{147}$$

This is a somewhat exaggerated value of the maximum dynamical deflection, because damping was completely neglected in the above discussion.

By using the principle of superposition the solution of the problem in the case of a system of concentrated moving forces and in the case of moving distributed forces can be also solved without difficulty.†

Moving Pulsating Force.‡—Consider now the case when a pulsating force is moving along the beam with a constant velocity v. Such a condition may occur, for instance, when an imperfectly balanced locomotive passes over a bridge (Fig. 120). The vertical component of the centrifugal force P, due to the unbalance, is § $P \cos \omega t_1$, where $\omega =$ the

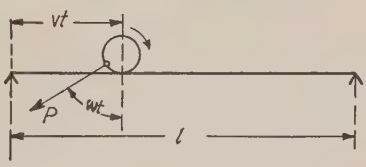


FIG. 120.

* The potential energy of the beam bent by the force P at the middle is

$$V = \frac{P^2l^3}{96EI} \quad \text{and} \quad \frac{W}{V} = 2.43.$$

This ratio is very close to the square of the ratio of the maximum deflections for the dynamical and statical conditions which is equal to $(48/\pi^3)^2 = 2.38$. The discrepancy should be attributed to the higher harmonics in the deflection curve.

† See writer's paper mentioned above.

‡ See writer's paper in Phil. Mag., Vol. 43 (1922), p. 1018.

§ It is assumed that at the initial moment $t_1 = 0$ the centrifugal force is acting in downwards direction.

angular velocity of the driving wheel. By using the same manner of reasoning as before, the following expression for the generalized force, corresponding to the generalized coordinate q_i will be obtained.

$$Q_i = P \cos \omega t_1 \sin \frac{i\pi v t_1}{l}.$$

Substituting this in the third term of the general solution (*c*), we obtain

$$y = \frac{Pl^3}{EI\pi^4} \sum_{i=1}^{i=\infty} \sin \frac{i\pi x}{l} \left(\frac{\sin\left(\frac{i\pi v}{l} + \omega\right)t}{i^4 - (\beta + i\alpha)^2} + \frac{\sin\left(\frac{i\pi v}{l} - \omega\right)t}{i^4 - (\beta - i\alpha)^2} - \frac{\alpha}{i}\left(\frac{\sin \frac{i^2\pi^2 a t}{l^2}}{-i^2\alpha^2 + (i^2 - \beta)^2} + \frac{\sin \frac{i^2\pi^2 a t}{l^2}}{-i^2\alpha^2 + (i^2 + \beta)^2} \right)\right) \quad (148)$$

where $\alpha = vl/a\pi =$ the ratio of the period $\tau = 2l^2/\pi a$ of the fundamental type of vibration of the beam to twice the time, $\tau_1 = l/v$, it takes the force P to pass over the beam,

$\beta = \tau/\tau_2 =$ the ratio of the period of the fundamental type of vibration of the beam to the period $\tau_2 = 2\pi/\omega$ of the pulsating force.

When the period τ_2 of the pulsating force is equal to the period τ of the fundamental type of vibration of the beam $\beta = 1$ and we obtain the condition of resonance. The amplitude of the vibration during motion of the pulsating force will be gradually built up and attains its maximum at the moment $t = l/v$ when the first term (for $i = 1$) in the series on the right of (148), which is the most important part of y, may be reduced to the form

$$\frac{1}{\alpha} \frac{2Pl^3}{EI\pi^4} \sin \frac{\pi x}{l} \sin \omega t$$

and the maximum deflection is given by the formula

$$\delta_{\max} = \frac{1}{\alpha} \frac{2Pl^3}{EI\pi^4} = \frac{2\tau_1}{\tau} \cdot \frac{2Pl^3}{EI\pi^4}. \quad (149)$$

Due to the fact that in actual cases the time interval $\tau_1 = l/v$ is large in comparison with the period τ of the natural vibration, the maximum dynamical deflection produced by the pulsating force P will be many times greater than the deflection $2Pl^3/EI\pi^4$, which would be produced by the same force if applied statically at the middle of the beam. Some

applications of eq. (149) for calculating the impact effect in bridges will be given in the next paragraph.

45. Vibration of Bridges.—It is well known that a rolling load produces in a bridge or in a girder a greater deflection and greater stresses than the same load acting statically. Such an "*impact effect*" of live loads on bridges is of great practical importance and many engineers have worked on the solution of this problem.* There are various causes producing impact effect on bridges of which the following will be discussed: (1) Live-load effect of a smooth-running load; (2) Impact effect of the balance-weights of the locomotive driving wheels and (3) Impact effect due to irregularities of the track and flat spots on the wheels.

Live-load Effect of a Smoothly Running Mass.—In discussing this problem two extreme cases will be considered: (1) when the mass of the moving load is large in comparison with the mass of the beam, i.e., girder or rail bearer, and (2) when the mass of the moving load is small in comparison with the mass of the bridge. In the first case the mass of the beam can be neglected. Then the deflection of the beam under the load at any position of this load will be proportional to the pressure R, which the rolling load P produces on the beam (Fig. 121) and can be calculated from the known equation of statical deflection:

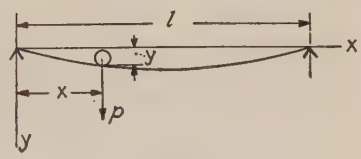


FIG. 121.

$$y = \frac{Rx^2(l - x)^2}{3lEI}. \qquad (a)$$

In order to obtain the pressure R the inertia force $-(P/g)(d^2y/dt^2)$ should be added to the rolling load P. Assuming that the load is moving along the beam with a constant velocity v, we obtain

$$\frac{dy}{dt} = v\frac{dy}{dx}; \qquad \frac{d^2y}{dt^2} = v^2\frac{d^2y}{dx^2}$$

and the pressure on the beam will be

$$R = P\left(1 - \frac{v^2}{g}\frac{d^2y}{dx^2}\right). \qquad (b)$$

* The history of the subject is extensively discussed in the famous book by Clebsch, Theorie der Elastizität fester Körper, traduite p. S. Venant (Paris 1883), see Note du par. 61, p. 597.

Substituting in eq. (a) we obtain

$$y = P\left(1 - \frac{v^2}{g}\frac{d^2y}{dx^2}\right)\frac{x^2(l-x)^2}{3lEI}. \tag{150}$$

This equation determines the trajectory of the point of contact of the rolling load with the beam.* An approximation of the solution of eq. (150) will be obtained by assuming that the trajectory is the same as at zero speed ($v = 0$) and by substituting

$$\frac{Px^2(l-x)^2}{3lEI}$$

for y in the right side of this equation. Then by simple calculations it can be shown that y becomes maximum when the load is at the middle of the span and the maximum pressure will be

$$R_{\max} = P\left(1 + \frac{v^2}{g}\frac{Pl}{3EI}\right). \tag{c}$$

The maximum deflection in the center of the beam increases in the same rate as the pressure on it, so that:

$$\delta_d = \delta_{st}\left(1 + \frac{v^2}{g}\frac{Pl}{3EI}\right). \tag{151}$$

This approximate solution as compared with the result of an exact solution of the eq. (150)† is accurate enough for practical applications. The additional term in the brackets is usually very small and it can be concluded that the "live-load effect" in the case of small girders has no practical importance.

In the second case when the mass of the load is small in comparison with the mass of the bridge the moving load can be replaced, with sufficient accuracy, by a moving force and then the results given in paragraph 44 can be used. Assuming, for instance, that for three single

* This equation was established by Willis: Appendix to the Report of the Commissioners . . . to inquire into the Application of Iron to Railway Structures (1849), London.

† The exact solution of eq. (150) was obtained by G. G. Stokes, see Math. and Phys. Papers, Vol. 2, p. 179. The same problem has been discussed also by H. Zimmermann, see Die Schwingungen eines Traegers mit bewegter Last. Berlin, 1896. It should be noted that the integration of eq. (150) can be made also numerically by using the method explained before, see p. 79. In this manner solutions for a beam on elastic supports and for continuous beams were obtained by Prof. N. P. Petroff, see, the Memoirs of The Russian Imperial Technical Society (1903).

track railway bridges with spans of 60 feet, 120 feet and 360 feet, the natural frequencies are as shown in the table below,*

	$l =$ 60 ft.	120 ft.	360 ft.
$f =$	9	5	2
$(\alpha)_{v=120 \text{ ft. per sec.}} =$	1/9	1/10	1/12

and taking the velocity $v = 120$ feet per sec., the quantity α, representing the ratio of the period of the fundamental type of vibration to twice the time l/v for the load to pass over the bridge will be as shown in the third line of the table. Now on the basis of solution (147) it can be concluded † that for a span of 60 feet and with a very high velocity, the increase in deflection due to the live load effect is about 12 per cent and this is still diminished with a decrease of velocity and with an increase of span. If several moving loads are acting on the bridge the oscillations associated with these should be superimposed. Only in the exceptional case of synchronism of these vibrations the resultant live-load effect on the system will be equal to the sum of the effects of the separate loads and the increase in deflection due to this effect will be in the same proportion as for a single load. From these examples it can be concluded that the live-load effect of a smooth-running load is not an important factor and in the most unfavorable cases it will hardly exceed 10 per cent. Much more serious effects may be produced, as we will see, by pulsating forces due to rotating balance weights of steam locomotives.

Impact Effect of Unbalanced Weights.—The most unfavorable condition occurs in the case of resonance when the number of revolutions per second of the driving wheels is equal to the frequency of natural vibration of the bridge. For a short span bridge the frequency of natural vibration is usually so high that synchronism of the pulsating load and the natural vibration is impossible at any practical velocity. By taking, for instance, six revolutions per second of the driving wheels as the highest limit and taking the frequencies of natural vibration from the table above it can be concluded that the resonance condition is hardly possible for spans less than 100 ft. For larger spans resonance conditions should

* Some experimental data on vibrations of bridges can be found in the following papers: A. Bühler, Stosswirkungen bei eisernen Eisenbahnbrücken, Druckschrift zum Intern. Kongress für Brückenbau, Zürich, 1926 and W. Hort, Stossbeanspruchungen und Schwingungen . . . . Die Bautechnik, 1928, Berlin.

† The bridge is considered here as a simple beam of a constant cross section. Vibration of trusses has been discussed by H. Reissner, Zeitschr. f. Baut., Vol. 53 (1903), p. 135, and E. Pohlhausen, Zeitschr. f. Angew. Math. u. Mech., Vol. 1 (1921), p. 28.

be taken into consideration and the impact effect should be calculated from eq. (149).

Let P_1 = the maximum resultant pressure on the rail due to the counterweights when the driving wheels are revolving once per second.

n = the total number of revolutions of the driving wheels during passage along the bridge.

Then, from eq. (149), we obtain the following additional deflection due to the impact effect,

$$\delta_{\max} = \frac{2n}{\tau^2}\frac{2P_1 l^3}{EI\pi^4}. \tag{152}$$

It is clear that in calculating the impact effect due to unbalanced weights we have to take consideration of: (1) the statical deflection produced by the force P_1, (2) the period τ of the natural vibration of the bridge and (3) the number of revolutions n. All these quantities are usually disregarded in impact formulas as applied in bridge design.

In order to obtain some idea about the amount of this impact effect let us apply eq. (152) to a numerical example * of a locomotive crossing a bridge of 120 feet span. Assuming that the locomotive load is equivalent to a uniform load of 14,700 lbs. per linear foot distributed over a length of 15 feet, and that the train load following and preceding the locomotive is equivalent to a uniformly distributed load of 5,500 lbs. per linear foot, the maximum central deflection of each girder is $(2l^3/EI\pi^4)(275{,}000)$ approximately. The same deflection when the locomotive approaches the support and the train completely covers the bridge is $(2l^3/EI\pi^4)(206{,}000)$ approximately. Taking the number of revolutions $n = 8$ (the diameter of the wheels equal to 4 feet and 9 inches) and the maximum pulsating pressure on each girder at the resonance condition equal to $P_1/\tau^2 = 18{,}750$ lbs., the additional deflection, calculated from eq. (152), will be $(2l^3/EI\pi^4)(300{,}000)$. Adding this to the statical deflection, calculated above for the case of the locomotive approaching the end of the bridge, we obtain for the complete deflection at the center $(2l^3/EI\pi^4)(506{,}000)$. Comparing this with the maximum statical central deflection $(2l^3/EI\pi^4) \times (275{,}000)$, given above, it can be concluded that the increase in deflection due to impact is in this case about 84 per cent. Assuming the number of revolutions n equal to 6 (the diameter of driving wheels equal to 6½ feet) and assuming again a condition of resonance, we will

* The figures below are taken from the paper by C. E. Inglis, previously mentioned (see p. 242).

obtain for the same numerical example an increase in deflection equal to 56 per cent.

In the case of bridges of shorter spans, when the frequency of natural vibration is considerably larger than the number of revolutions per second of the driving wheels, a satisfactory approximation can be obtained by taking only the first term in the series (148) and assuming the most unfavorable condition, namely, that $\sin([\pi v/l] + \omega)t$ and $\sin([\pi v/l] - \omega)t$ become equal to 1 and $\sin \pi^2 at/l^2$ equal to -1 at the moment $t = l/2v$ when the pulsating force arrives at the middle of the span. Then the additional deflection, from (148), will be

$$\begin{aligned} &= \frac{Pl^3}{EI\pi^4}\left(\frac{1}{1-(\beta+\alpha)^2} + \frac{1}{1-(\beta-\alpha)^2} + \frac{\alpha}{(1-\beta)^2-\alpha^2} + \frac{\alpha}{(1+\beta)^2-\alpha^2}\right) \\ &= \frac{2Pl^3}{EI\pi^4}\frac{1-\alpha}{(1-\beta[1+\alpha/\beta])(1+\beta[1-\alpha/\beta])}. \end{aligned} \tag{153}$$

Consider, for instance, a 60-foot span bridge and assume the same kind of loading as in the previous example, then the maximum statical deflection is $(2l^3/EI\pi^4)(173{,}000)$ approximately. If the driving wheels have a circumference of 20 feet and make 6 revolutions per second, the maximum downwards force on the girder will be $18{,}750(6/5)^2 = 27{,}000$ lbs. Assuming the natural frequency of the bridge equal to 9, we obtain, from eq. (153)

$$\delta = \frac{2l^3}{EI\pi^4}(27{,}000 \times 2.57) = \frac{2l^3}{EI\pi^4}(69{,}400).$$

Hence,

$$\frac{\text{dynamical deflection}}{\text{statical deflection}} = \frac{173 + 69.4}{173} = 1.40.$$

The impact effect of the balancing weights in this case amounts to 40 per cent.

In general it will be seen from the theory developed above that the most severe impact effects will be obtained in the shortest spans for which a resonance condition may occur (about 100 feet spans for the assumption made above) because in this case the resonance occurs when the pulsating disturbing force has its greatest magnitude. With increase in the span the critical speed decreases and also the magnitude of the pulsating load, consequently the impact effect decreases. For very large spans, when the frequency of the fundamental type of vibration is low, synchronism of the pulsating force with the second mode of vibration having a node at the middle of the span becomes theoretically possible and due to this

cause an increase in the impact effect may occur at a velocity of about four times as great as the first critical speed.

It should be noted that all our calculations were based on the assumption of a pulsating force moving along the bridge. In actual conditions we have rolling *masses*, which will cause a variation in the natural frequency of the bridge in accordance with the varying position of the loads. This variability of the natural frequency which is especially pronounced in short spans is very beneficial because the pulsating load will no longer be in resonance all the time during passing over the bridge and its cumulative effect will not be as pronounced as is given by the above theory. From experiments made by the Indian Railway Bridge Committee,* it is apparent that on the average the maximum deflection occurs when the engine has traversed about two-thirds of the span and that the maximum impact effect amounts to only about one-third of that given by eq. (152). It should be noted also that the impact effect is proportional to the force P_1 and depends therefore on the type of engine and on the manner of balancing. While in a badly balanced two cylinder engine the force P_1 may amount to more than 1000 lbs.,† in electric locomotives, perfect balancing can be obtained without introducing a fluctuating rail pressure. This absence of impact effect may compensate for the increase in axle load in modern heavy electric locomotives.

In the case of short girders and rail bearers whose natural frequencies are very high, the effect of counter-weights on the deflection and stresses can be calculated with sufficient accuracy by neglecting vibrations and using the statical formula in which the centrifugal forces of the counter-weights should be added to the statical rail pressures. The effect of these centrifugal forces may be especially pronounced in the case of short spans when only a small number of wheels can be on the girder simultaneously.

Impact Effects Due to Irregularities of Track and Flats on Wheels.—Irregularities like low spots on the rails, rail joints, flats on the wheels, etc., may be responsible for considerable impact effect which may become especially pronounced in the case of short spans. If the shape of the low spots in the track or of the flats on the wheels is given by a smooth curve, the method used before in considering the effect of road unevenness on the vibrations of vehicles (see p. 136) can also be applied here for calcu-

* See Bridge Sub-Committee Reports, 1925; Calcutta: Government of India Central Publication Branch, Technical Paper No. 247 (1926).

† Some data on the values of P_1 for various types of engines are given in the Bridge Sub-Committee Report, mentioned above.

lating the additional pressure of the wheel on the rail. This additional pressure will be proportional to the unsprung mass of the wheel and to the square of the velocity of the train. It may attain a considerable magnitude and has practical importance in the case of short bridges and rail bearers. This additional dynamical effect produced by irregularities in the track and flats on the wheels justifies the high impact factor usually applied in the design of short bridges. By removing rail joints from the bridges and by using ballasted spans or those provided with heavy timber floors, the effect of these irregularities can be diminished and the strength condition considerably improved.

46. Effect of Axial Forces on Lateral Vibrations. — *Bar with Hinged Ends.*—As a first example of this type of vibration let us consider the case of a bar compressed by two forces S (see Fig. 122). The general expression for the lateral vibration will be the same as before (see eq. (138))

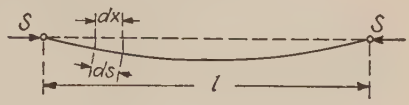


FIG. 122.

$$y = \sum_{i=1}^{i=\infty} q_i \sin \frac{i\pi x}{l} \cdot \qquad (a)$$

The difference will be only in the expression for the potential energy of the system. It will be appreciated that during lateral deflection in this case not only the energy of bending but also the change in the energy of compression should be considered. Due to lateral deflection the initially compressed center line of the bar expands somewhat * and the potential energy of compression diminishes. The increase in length of the center line will be (see Fig. 122),

$$\int_0^l (ds - dx) = \tfrac{1}{2} \int_0^l \left(\frac{dy}{dx}\right)^2 dx.$$

The corresponding diminishing of the energy of compression is †

$$\frac{S}{2}\int_0^l \left(\frac{dy}{dx}\right)^2 dx = \frac{S}{2}\int_0^l \left(\sum_{i=1}^{i=\infty} q_i \frac{i\pi}{l} \cos\frac{i\pi x}{l}\right)^2 dx = \frac{S\pi^2}{4l}\sum_{i=1}^{i=\infty} i^2 q_i^2. \qquad (b)$$

If the ends of the bar are free to slide in an axial direction eq. (b) will represent the work of forces S. For the energy of bending the equation (143) previously obtained will be used. Hence the complete potential

* The hinges are assumed immovable during vibration.

† Only those deflections are considered here which are so small that any change in longitudinal force can be neglected.

energy becomes

$$V = \frac{EI\pi^4}{4l^3}\sum_{i=1}^{i=\infty} i^4 q_i^2 - \frac{S\pi^2}{4l}\sum_{i=1}^{i=\infty} i^2 q_i^2. \tag{154}$$

The kinetic energy of the bar, from eq. (144) is

$$T = \frac{\gamma A l}{4g}\sum_{i=1}^{i=\infty} \dot{q}_i^2. \tag{155}$$

Now Lagrange's equation for any coordinate q_i will be

$$\frac{\gamma A l}{2g}\ddot{q}_i + \frac{EI\pi^4}{2l^3}\left(i^4 - \frac{Sl^2}{EI\pi^2}i^2\right)q_i = 0.$$

By using the notations,

$$a^2 = \frac{EIg}{\gamma A}, \qquad \alpha^2 = \frac{Sl^2}{EI\pi^2}, \tag{156}$$

we obtain

$$\ddot{q}_i + \frac{a^2\pi^4}{l^4}(i^4 - \alpha^2 i^2)q_i = 0,$$

from which,

$$q_i = C\cos\left(\frac{a\pi^2 i^2}{l^2}\sqrt{1 - \frac{\alpha^2}{i^2}}\right)t + D\sin\left(\frac{a\pi^2 i^2}{l^2}\sqrt{1 - \frac{\alpha^2}{i^2}}\right)t. \tag{157}$$

Substituting this in (*a*) the complete expression for free vibrations will be obtained.

Comparing this solution (157) with (136) it can be concluded that, due to the compressive force S, the frequencies of natural vibration are diminished in the proportion

$$\sqrt{1 - \frac{\alpha^2}{i^2}}.$$

If α^2 approaches 1, the frequency of the fundamental type of vibration approaches zero, because at this value of α^2 the compressive force S attains its critical value $EI\pi^2/l^2$ where the straight form of equilibrium becomes unstable and the bar buckles sidewise.

If instead of a compressive a tensile force S is acting on the bar the frequency of vibration increases. In order to obtain the free vibrations in this case it is only necessary to change the sign of α^2 in eq. (157). Then

$$q_i = C\cos\left(\frac{a\pi^2 i^2}{l^2}\sqrt{1 + \frac{\alpha^2}{i^2}}\right)t + D\sin\left(\frac{a\pi^2 i^2}{l^2}\sqrt{1 + \frac{\alpha^2}{i^2}}\right)t. \tag{157^1}$$

When α^2 is a very large number (such conditions can be obtained with thin wires or strings) 1 can be neglected in comparison with α^2/i^2 and we obtain from (157)[1]

$$q_i = C \cos \frac{i\pi}{l} \sqrt{\frac{gS}{A\gamma}}\, t + D \sin \frac{i\pi}{l} \sqrt{\frac{gS}{A\gamma}}\, t.$$

Substituting in (*a*)

$$y = \sum_{i=1}^{i=\infty} \sin \frac{i\pi x}{l} \left(C \cos \frac{i\pi}{l} \sqrt{\frac{gS}{A\gamma}}\, t + D \sin \frac{i\pi}{l} \sqrt{\frac{gS}{A\gamma}}\, t \right). \tag{158}$$

This is the general solution for the lateral vibrations of a stretched string where the rigidity of bending is neglected.

Cantilever Beam.—In this case only an approximate solution, by using the Rayleigh method, will be given. As a basis of this calculation the deflection curve

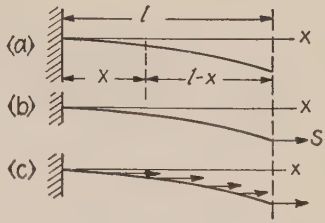


FIG. 123.

$$y = \frac{q}{2EI} \left(\frac{1}{2} l^2 x^2 - \frac{1}{3} l x^3 + \frac{1}{12} x^4 \right) \tag{c}$$

of a cantilever under the action of its weight q per unit length will be taken. The potential energy of bending in this case is

$$V = \frac{EI}{2} \int_0^l \left(\frac{d^2y}{dx^2} \right)^2 dx = \frac{q^2 l^5}{40EI}. \tag{d}$$

If the deflection during vibration is given by $y \cos pt$, the maximum kinetic energy of vibration will be

$$T = \frac{qp^2}{2g} \int_0^l y^2 dx = \frac{q^2 p^2 l^9}{E^2 I^2 g} \frac{13}{8 \cdot 9 \cdot 90}. \tag{e}$$

Putting (*d*) equal to (*e*) the following expression for the frequency and the period of vibration of a cantilever (Fig. 123, *a*) will be obtained

$$f = \frac{p}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{90 \times 9}{65}} \sqrt{\frac{EIg}{ql^4}}, \tag{159}$$

$$\tau = \frac{1}{f} = 2\pi \sqrt{\frac{65}{90 \times 9}} \sqrt{\frac{ql^4}{EIg}} = \frac{2\pi}{3.530} \sqrt{\frac{ql^4}{EIg}}. \tag{160}$$

The error of this approximate solution is less than ½ per cent (see p. 234).

In order to calculate the frequency when a tensile force S is acting at the end of the cantilever, the quantity given by (f)

$$\frac{S}{2}\int_0^l \left(\frac{dy}{dx}\right)^2 dx = \frac{Sq^2l^7}{8 \times 14 \times E^2I^2}, \tag{f}$$

which is equal and opposite in sign to the work done by the tensile force S during bending, should be added to the potential energy of bending, calculated above (eq. (d)). Then

$$V = \frac{q^2l^5}{40EI}\left(1 + \frac{5}{14}\frac{Sl^2}{EI}\right). \tag{g}$$

Due to this increase in potential energy the frequency of vibration will be found by multiplying the value (159) by

$$\sqrt{1 + \frac{5}{14}\frac{Sl^2}{EI}}. \tag{161}$$

It is interesting to note that the term $5/14\ Sl^2/EI$ differs only about 10 per cent from the quantity $\alpha^2 = 4Sl^2/EI\pi^2$, representing the ratio of the longitudinal force S to the critical column force for a cantilever.

If tensile forces s are uniformly distributed along the length of the cantilever (Fig. 123, c), the term to be added to the energy of bending will be

$$\tfrac{1}{2}\int_0^l \left(\frac{dy}{dx}\right)^2 s(l - x)dx = \frac{q^2l^7}{8 \times 14E^2I^2}\frac{7sl}{20}. \tag{162}$$

Comparing with eq. (f) it can be concluded that the effect on the frequency of uniformly distributed tensile forces is the same as if 7/20 of the sum of these forces be applied at the end of the cantilever.

This result may be of some practical interest in discussing the effect of the centrifugal force on the frequency of vibration of turbine blades (see p. 272).

47. Vibration of Beams on Elastic Foundation.—Assume that a beam with hinged ends is supported along its length by a *continuous elastic foundation*, the rigidity of which is given by the magnitude k of the *modulus of foundation*. k is the load per unit length of the beam necessary to produce a compression in the foundation equal to unity. If the mass of the foundation can be neglected the vibrations of such a beam can easily be studied by using the same methods as before. It is only necessary in calculating the potential energy of the system to add to the energy of bending of the beam, the energy of deformation of the elastic

foundation. Taking, as before, for hinged ends,

$$y = \sum_{i=1}^{i=\infty} q_i \sin \frac{i\pi x}{l}, \tag{a}$$

we obtain

$$V = \frac{EI}{2}\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx + \frac{k}{2}\int_0^l y^2 dx = \frac{EI\pi^4}{4l^3}\sum_{i=1}^{i=\infty} i^4 q_i^2 + \frac{kl}{4}\sum_{i=1}^{i=\infty} q_i^2 \cdots \tag{163}$$

The first series in this expression represents the energy of bending of the beam (see eq. 143) and the second series the energy of deformation of the foundation.

The kinetic energy of vibration is, from eq. (144),

$$T = \frac{\gamma Al}{4g}\sum_{i=1}^{i=\infty} \dot{q}_i^2.$$

The differential equation of motion for any coordinate q_i is

$$\ddot{q}_i + \frac{2g}{\gamma Al}\left(\frac{EI\pi^4}{2l^3} i^4 + \frac{kl}{2}\right) q_i = \frac{2g}{\gamma Al} Q_i$$

or

$$\ddot{q}_i + \frac{a^2\pi^4}{l^4}(i^4 + \beta) q_i = \frac{2g}{\gamma Al} Q_i. \tag{b}$$

in which Q_i denotes the external disturbing force corresponding to the coordinate q_i

$$a^2 = \frac{EIg}{\gamma A}; \qquad \beta = \frac{kl^4}{EI\pi^4}. \tag{164}$$

By taking $\beta = 0$, the equation for a hinged bar unsupported by any elastic foundation will be obtained (see p. 238). Denoting,

$$p_i^2 = \frac{a^2\pi^4}{l^4}(i^4 + \beta). \tag{c}$$

a general solution of equation (*b*) will be

$$q_i = A_i \cos p_i t + B_i \sin p_i t + \frac{2g}{\gamma Al}\cdot\frac{1}{p_i}\int_0^t Q_i \sin p_i(t - t_1) dt_1. \tag{d}$$

The two first terms of this solution represent free vibrations of the beam, depending on the initial conditions. The third term represents vibrations produced by the disturbing force Q_i.

The frequencies of the natural vibrations depend, as seen from (*c*), not only on the rigidity of the beam but also on the rigidity of the foundation.

As an example consider the case when a pulsating force $P = P_0 \sin nt$ is acting on the beam at a distance c from the left support (Fig. 118). The generalized force corresponding to the coordinate q_i will be in this case

$$Q_i = P_0 \sin \frac{i\pi c}{l} \sin nt_1. \tag{e}$$

Substituting in eq. (d) and considering only vibrations produced by the disturbing force we obtain

$$q_i = \frac{2g}{\gamma A} P_0 \sin \frac{i\pi c}{l} \left\{ \frac{l^3}{\pi^4 a^2 (i^4 + \beta) - n^2 l^4} \sin nt - \frac{n}{l p_i (p_i^2 - n^2)} \sin p_i t \right\}.$$

Substituting in (a)

$$y = \frac{2gP_0 l^3}{\gamma A} \sum_{i=1}^{i=\infty} \left\{ \frac{\sin \frac{i\pi c}{l} \sin \frac{i\pi x}{l} \sin nt}{\pi^4 a^2 (i^4 + \beta) - n^2 l^4} - \frac{n \sin \frac{i\pi c}{l} \sin \frac{i\pi x}{l} \sin p_i t}{l^4 p_i (p_i^2 - n^2)} \right\}. \tag{f}$$

The first term in this expression represents the forced vibration and the second, the free vibration of the beam. By taking $n = 0$ and $P = P_0 \sin nt$ the deflection of the beam by a constant force P will be obtained:

$$y = \frac{2Pl^3}{EI\pi^4} \sum_{i=1}^{i=\infty} \frac{\sin \frac{i\pi c}{l} \sin \frac{i\pi x}{l}}{i^4 + \beta}. \tag{165}$$

By taking $c = l/2$ the deflection by the force P at the middle will be obtained as below:

$$y = \frac{2Pl^3}{EI\pi^4} \left(\frac{\sin \frac{\pi x}{l}}{1 + \beta} - \frac{\sin \frac{3\pi x}{l}}{3^4 + \beta} + \frac{\sin \frac{5\pi x}{l}}{5^4 + \beta} - \cdots \right). \tag{166}$$

Comparing this with eq. (h), p. 240, it can be concluded that the additional term β in the denominators represents the effect of the elastic foundation on the deflection of the beam.

By comparing the forced vibrations

$$y = \frac{2gP_0 l^3 \sin nt}{\gamma A} \sum_{i=1}^{i=\infty} \frac{\sin \frac{i\pi c}{l} \sin \frac{i\pi x}{l}}{\pi^4 a^2 (i^4 + \beta) - n^2 l^4} = \frac{2Pl^3}{EI\pi^4} \sum_{i=1}^{i=\infty} \frac{\sin \frac{i\pi c}{l} \sin \frac{i\pi x}{l}}{i^4 + \beta - \frac{n^2 l^4}{\pi^4 a^2}},$$

with the statical deflection (165) it can be concluded that the dynamical

deflections can be obtained from the statical formula. It is only necessary to replace β by $\beta - (n^2l^4/\pi^4a^2)$.

By using the notations (164), we obtain

$$\beta - \frac{n^2l^4}{\pi^4a^2} = \frac{kl^4}{EI\pi^4} - \frac{n^2l^4\gamma A}{\pi^4EIg} = \frac{l^4}{EI\pi^4}\left(k - \frac{\gamma n^2A}{g}\right).$$

This means that the dynamical deflection can be obtained from the statical formula by replacing in it the actual modulus of foundation by a diminished value $k - (\gamma n^2A/g)$ of the same modulus. This conclusion remains true also in the case of an infinitely long bar on an elastic foundation. By using it the deflection of a rail produced by a pulsating load can be calculated.*

48. Ritz Method.†—It has already been shown in several cases in previous chapters (see paragraph 14) that in calculating the frequency of the fundamental type of vibration of a complicated system the approximate method of Rayleigh can be applied. In using this method it is necessary to make some assumption as to the shape of the deflection curve of a vibrating beam or vibrating shaft. The corresponding frequency will then be found from the consideration of the energy of the system. The choosing of a definite shape for the deflection curve in this method is equivalent to introducing some additional constraints which reduces the system to one having a single degree of freedom. Such additional constraints can only increase the rigidity of the system and make the frequency of vibration, as obtained by Rayleigh's method, always somewhat higher than its exact value. Better approximations in calculating the fundamental frequency and also the frequencies of higher modes of vibration can be obtained by Ritz's method which is a further development of Rayleigh's method. In using this method the deflection curve representing the mode of vibration is to be taken with several parameters, the magnitudes of which should be chosen in such a manner as to reduce to a minimum the frequency of the fundamental type of vibration. The manner of choosing the shape of the deflection curve and the procedure of calculating consecutive frequencies will now be shown for the simple case of the vibration of a uniform string (Fig. (124). Let

* See writer's paper, Statical and Dynamical Stresses in Rails, Intern. Congress for Applied Mechanics, Proceedings, Zürich, 1926, p. 407.

† See Walther Ritz, Gesammelte Werke, p. 265 (1911), Paris.

S = tensile force in the string,

q = the weight of the string per unit length,

$2l$ = the length of the string.

If the string performs one of the normal modes of vibration, the deflection can be represented as follows:

$$y = X \cos pt, \tag{a}$$

where X is a function of x determining the shape of the vibrating string, and p determines the frequency of vibration. Assuming that the deflections are very small, the change in the tensile force S during vibration can be neglected and the increase in potential energy of deformation due to the deflection will be obtained by multiplying S with the increase in length of the string. In this manner the following expression for the potential energy is found, the energy in the position of equilibrium being taken as zero,

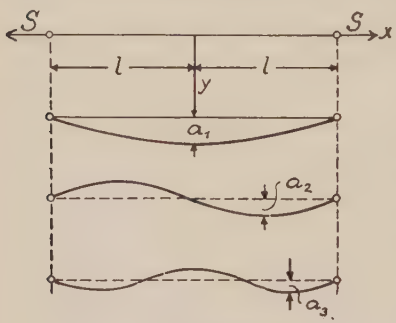


FIG. 124.

$$V = S \int_0^l \left(\frac{dy}{dx}\right)^2 dx.$$

The maximum potential energy occurs when the vibrating string occupies its extreme position. In this position $\cos pt = 1$ and

$$V = S \int_0^l \left(\frac{dX}{dx}\right)^2 dx. \tag{b}$$

The kinetic energy of the vibrating string is

$$T = \frac{q}{g} \int_0^l (\dot{y})^2 dx.$$

Its maximum occurs when the vibrating string is in its middle position, i.e., when $\cos pt = 0$, then

$$T = \frac{p^2 q}{g} \int_0^l X^2 dx. \tag{c}$$

Assuming that there are no losses in energy, we may equate (b) and

(*c*), thus obtaining

$$p^2 = \frac{gS}{q} \frac{\int_0^l \left(\frac{dX}{dx}\right)^2 dx}{\int_0^l X^2 dx}. \tag{d}$$

Knowing the shapes of various modes of vibration and substituting in (*d*) the corresponding expressions for X, the frequencies of these modes of vibration can easily be calculated. In the case of a uniform string, the deflection curves during vibration are sinusoidal curves and for the first three modes of vibration, shown in Fig. 124, we have

$$X_1 = a_1 \cos\frac{\pi x}{2l}; \qquad X_2 = a_2 \sin\frac{\pi x}{l}; \qquad X_3 = a_3 \cos\frac{3\pi x}{2l}.$$

Substituting in (*d*) we obtain (See eq. 158)

$$p_1^2 = \frac{\pi^2}{4l^2}\frac{gS}{q}, \qquad p_2^2 = \frac{\pi^2}{l^2}\frac{gS}{q}, \qquad p_3^2 = \frac{9}{4}\frac{\pi^2}{l^2}\frac{gS}{q}, \tag{e}$$

and the corresponding frequencies will be

$$f_1 = \frac{p_1}{2\pi} = \frac{1}{4l}\sqrt{\frac{gS}{q}}; \qquad f_2 = \frac{2}{4l}\sqrt{\frac{gS}{q}}; \qquad f_3 = \frac{3}{4l}\sqrt{\frac{gS}{q}}. \tag{f}$$

Let us now apply Ritz's method in calculating from eq. (*d*) the frequency f_1 of the fundamental type of vibration. The first step in the application of this method is the choosing of a suitable expression for the deflection curve. Let $\varphi_1(x)$, $\varphi_2(x)$, $\cdots$ be a series of functions satisfying the end conditions and suitable for representation of X. Then, by taking

$$X = a_1\varphi_1(x) + a_2\varphi_2(x) + a_3\varphi_3(x) + \cdots, \tag{g}$$

we can obtain a suitable deflection curve of the vibrating string.

We know that by taking a finite number of terms in the expression (*g*) we superimpose certain limitations on the possible shapes of the deflection curve of the string and due to this fact the frequency, as calculated from (*d*), will usually be higher than the exact value of this frequency. In order to obtain the approximation as close as possible, Ritz proposed to choose the coefficients a_1, a_2, a_3, $\cdots$ in the expression (*g*) so as to make the expression (*d*) a minimum. In this manner a system of equations such

as

$$\frac{\partial}{\partial a_n} \frac{\int_0^l \left(\frac{dX}{dx}\right)^2 dx}{\int_0^l X^2 dx} = 0 \tag{h}$$

will be obtained.

Performing the differentiation indicated we have,

$$\int_0^l X^2 dx \frac{\partial}{\partial a_n} \int_0^l \left(\frac{dX}{dx}\right)^2 dx - \int_0^l \left(\frac{dX}{dx}\right)^2 dx \frac{\partial}{\partial a_n} \int_0^l X^2 dx = 0 \tag{k}$$

or noting from (d), that

$$\int_0^l \left(\frac{dX}{dx}\right)^2 dx = \frac{p^2 q}{gS} \int_0^l X^2 dx$$

we obtain, from (k)

$$\frac{\partial}{\partial a_n} \int_0^l \left\{ \left(\frac{dX}{dx}\right)^2 - \frac{p^2 q}{gS} X^2 \right\} dx = 0. \tag{l}$$

In this way a system of equations homogeneous and linear in a_1, a_2, a_3, $\cdots$ will be obtained, the number of which will be equal to the number of coefficients a_1, a_2, a_3, $\cdots$ in the expression (g). Such a system of equations can yield for a_1, a_2, a_3, $\cdots$ solutions different from zero only if the determinant of these equations is equal to zero. This condition brings us to the *frequency equation* from which the frequencies of the various modes of vibrations can be calculated.

Let us consider the modes of vibration of a taut string symmetrical with respect to the middle plane. It is easy to see that a function like as $l^2 - x^2$, representing a symmetrical parabolic curve and satisfying end conditions $\{(y)_{x=\pm l} = 0\}$ is a suitable function in this case. By multiplying this function with x^2, x^4, $\cdots$ a series of curves symmetrical and satisfying the end conditions will be obtained. In this manner we arrive at the following expression for the deflection curve of the vibrating string

$$X = a_1(l^2 - x^2) + a_2 x^2 (l^2 - x^2) + a_3 x^4 (l^2 - x^2) + \cdots. \tag{m}$$

In order to show how quickly the accuracy of our calculations increases with an increase in the number of terms of the expression (m) we begin with one term only and put

$$X_1 = a_1(l^2 - x^2).$$

Then,

$$\int_0^l (X_1)^2 dx = \frac{8}{15} a_1^2 l^5; \qquad \int_0^l \left(\frac{dX_1}{dx}\right)^2 dx = \frac{4}{3} a_1^2 l^3.$$

Substituting in eq. (d) we obtain

$$p_1^2 = \frac{5}{2l^2} \frac{gS}{q}.$$

Comparing this with the exact solution (e) it is seen that 5/2 instead of $\pi^2/4$ is obtained, and the error in frequency is only .66%.

It should be noted that by taking only one term in the expression (m) the shape of the curve is completely determined and the system is reduced to one with a single degree of freedom, as is done in Rayleigh's approximate method.

In order to get a further approximation let us take two terms in the expression (m). Then we will have two parameters a_1 and a_2 and by changing the ratio of these two quantities we can change also, to a certain extent, the shape of the curve. The best approximation will be obtained when this ratio is such that the expression (d) becomes a minimum, which is accomplished when the conditions (l) are satisfied.

By taking

$$X_2 = a_1(l^2 - x^2) + a_2 x^2 (l^2 - x^2)$$

we obtain

$$\int_0^l X_2^2 dx = \frac{8}{15} a_1^2 l^5 + \frac{16}{105} a_1 a_2 l^7 + \frac{8}{315} a_2^2 l^9,$$

$$\int_0^l \left(\frac{dX_2}{dx}\right)^2 dx = \frac{4}{3} a_1^2 l^3 + \frac{8}{15} a_1 a_2 l^5 + \frac{44}{105} a_2^2 l^7.$$

Substituting in eq. (l) and taking the derivatives with respect to a_1 and a_2 we obtain

$$\begin{aligned} a_1(1 - 2/5k^2l^2) + a_2 l^2(1/5 - 2/35k^2l^2) &= 0, \\ a_1(1 - 2/7k^2l^2) + a_2 l^2(11/7 - 2/21k^2l^2) &= 0, \end{aligned} \qquad (n)$$

in which

$$k^2 = \frac{p^2 q}{gS}. \qquad (p)$$

The determinant of the equations (n) will vanish when

$$k^4 l^4 - 28k^2 l^2 + 63 = 0.$$

The two roots of this equation are

$$k_1^2 l^2 = 2.46744, \qquad k_2^2 l^2 = 25.6.$$

Remembering that we are considering only modes of vibration symmetrical about the middle and using eq. (p) we obtain for the first and third modes of vibration,

$$p_1^2 = \frac{2.46744}{l^2}\frac{gS}{q}, \qquad p_3^2 = \frac{25.6}{l^2}\frac{gS}{q}.$$

Comparing this with the exact solutions (e)

$$p_1^2 = \frac{\pi^2}{4l^2}\frac{gS}{q} = \frac{2.467401}{l^2}\frac{gS}{q}; \qquad p_3^2 = \frac{9}{4}\frac{\pi^2}{l^2}\frac{gS}{q} = \frac{22.207}{l^2}\frac{gS}{q},$$

it can be concluded that the accuracy with which the fundamental frequency is obtained is very high (the error is less than .001%). The error in the frequency of the third mode of vibration is about 6.5%. By taking three terms in the expression (m) the frequency of the third mode of vibration will be obtained with an error less than ½%.*

It is seen that by using the Ritz method not only the fundamental frequency but also frequencies of higher modes of vibration can be obtained with good accuracy by taking a sufficient number of terms in the expression for the deflection curve. In the next paragraph an application of this method to the study of the vibrations of beams of variable cross section will be shown.

49. Vibration of Rods of Variable Cross Section.—*General.*—In our previous discussion various problems involving the vibration of prismatical bars were considered. There exist, however, several important engineering problems such as the vibration of turbine blades, of hulls of ships, of beams of variable depth, etc., in which recourse has to be taken to the theory of vibration of a rod of variable section. The differential equation of vibration of such a rod has been previously discussed (see p. 222) and has the following form,

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right) + \frac{A\gamma}{g}\frac{\partial^2 y}{\partial t^2} = 0, \qquad (115)^1$$

in which I and A are certain functions of x. Only in some special cases which will be considered later, the exact forms of the normal functions can be determined in terms of known functions and usually in the solution of such problems approximate methods like the Rayleigh-Ritz method are used for calculating the natural frequencies of vibration. By taking the deflection of the rod, while vibrating, in the form

$$y = X \cos pt, \qquad (a)$$

* See W. Ritz, mentioned above, p. 259.

in which X determines the *mode of vibration*, we obtain the following expressions for the maximum potential and the maximum kinetic energy,

$$V = \frac{1}{2}\int_0^l EI\left(\frac{d^2X}{dx^2}\right)^2 dx, \tag{b}$$

$$T = \frac{p^2}{2g}\int_0^l A\gamma X^2 dx, \tag{c}$$

from which

$$p^2 = \frac{Eg}{\gamma}\frac{\int_0^l I\left(\frac{d^2X}{dx^2}\right)^2 dx}{\int_0^l AX^2 dx}. \tag{d}$$

The exact solution for the frequency of the fundamental mode of vibration will be the one which makes the left side of (*d*) a minimum. In order to obtain an approximate solution we proceed as in the previous paragraph and take the shape of the deflection curve in the form of a series,

$$X = a_1\varphi_1(x) + a_2\varphi_2(x) + a_3\varphi_3(x) + \cdots, \tag{e}$$

in which every one of the functions φ satisfies the conditions at the end of the rod. Substituting (*e*) in eq. (*d*) the conditions of minimum will be

$$\frac{\partial}{\partial a_n}\frac{\int_0^l I\left(\frac{d^2X}{dx^2}\right)^2 dx}{\int_0^l AX^2 dx} = 0 \tag{f}$$

or

$$\int_0^l AX^2 dx \frac{\partial}{\partial a_n}\int_0^l I\left(\frac{d^2X}{dx^2}\right)^2 dx - \\ - \int_0^l I\left(\frac{d^2X}{dx^2}\right)^2 dx \frac{\partial}{\partial a_n}\int_0^l AX^2 dx = 0. \tag{g}$$

From (*g*) and (*d*) we obtain

$$\frac{\partial}{\partial a_n}\int_0^l \left(I\left(\frac{d^2X}{dx^2}\right)^2 - \frac{p^2A\gamma}{Eg}X^2\right) dx = 0. \tag{167}$$

The problem reduces to finding such values for the constants $a_1, a_2, a_3, \cdots$ in eq. (*e*) as to make the integral

$$S = \int_0^l \left(I\left(\frac{d^2X}{dx^2}\right)^2 - \frac{p^2A\gamma}{Eg}X^2\right) dx \tag{h}$$

a minimum.

The equations (167) are homogeneous and linear in $a_1, a_2, a_3, \cdots$, and their number is equal to the number of terms in the expression (e). Equating to zero the determinant of these equations, the *frequency equation* will be obtained from which the frequencies of the various modes can be calculated.

Fig. 125.

Vibration of a Wedge.—In the case of a wedge of constant unit thickness with one end free, and the other one built in (Fig. 125) we have

$$A = \frac{2bx}{l},$$

$$I = \frac{1}{12}\left(\frac{2bx}{l}\right)^3,$$

where l = length of the cantilever,

$2b$ = depth of the cantilever at the built-in end.

The end conditions are:

(1) $\left(EI\dfrac{d^2X}{dx^2}\right)_{x=0} = 0,$ (2) $\left(EI\dfrac{d^3X}{dx^3}\right)_{x=0} = 0,$

(3) $(X)_{x=l} = 0,$ (4) $\left(\dfrac{dX}{dx}\right)_{x=l} = 0.$

The first two conditions will be always satisfied because $I = 0$ for $x = 0$. In order to satisfy the conditions at the built-in end we take the deflection curve in the form of the series

$$X = a_1\left(1 - \frac{x}{l}\right)^2 + a_2\frac{x}{l}\left(1 - \frac{x}{l}\right)^2 + a_3\frac{x^2}{l^2}\left(1 - \frac{x}{l}\right)^2 + \cdots. \qquad (k)$$

It is easy to see that each term as well as its derivative with respect to x, becomes equal to zero when $x = l$. Consequently the end conditions (3) and (4) above will be satisfied.

Taking as a first approximation

$$X_1 = a_1\left(1 - \frac{x}{l}\right)^2$$

and substituting in (d) we obtain

$$p^2 = 10\frac{Eg}{\gamma}\frac{b^2}{l^4}; \qquad f = \frac{p}{2\pi} = \frac{5.48}{2\pi}\frac{b}{l^2}\sqrt{\frac{Eg}{3\gamma}}. \qquad (l)$$

In order to get a closer approximation we take two terms in (k), then

$$X_2 = a_1\left(1 - \frac{x}{l}\right)^2 + a_2\frac{x}{l}\left(1 - \frac{x}{l}\right)^2.$$

Substituting in (h)

$$S_2 = \frac{2}{3}\frac{b^3}{l^3}\left((a_1 - 2a_2)^2 + \frac{24}{5}a_2(a_1 - 2a_2) + 6a_2^2\right)$$
$$- \frac{2b\gamma l p^2}{Eg}\left(\frac{a_1^2}{30} + \frac{2a_1a_2}{105} + \frac{a_2^2}{280}\right).$$

Now from the conditions

$$\frac{\partial S_2}{\partial a_1} = 0, \qquad \frac{\partial S_2}{\partial a_2} = 0,$$

we obtain

$$\left(\frac{Eg}{\gamma}\frac{b^2}{3l^4} - \frac{p^2}{30}\right)a_1 + \left(\frac{2}{5}\frac{Eg}{\gamma}\frac{b^2}{3l^4} - \frac{p^2}{105}\right)a_2 = 0,$$
$$\left(\frac{2}{5}\frac{Eg}{\gamma}\frac{b^2}{3l^4} - \frac{p^2}{105}\right)a_1 + \left(\frac{2}{5}\frac{Eg}{\gamma}\frac{b^2}{3l^4} - \frac{p^2}{280}\right)a_2 = 0.$$

Equating to zero the determinant of these equations we get

$$\left(\frac{Eg}{\gamma}\frac{b^2}{3l^4} - \frac{p^2}{30}\right)\left(\frac{2}{5}\frac{Eg}{\gamma}\frac{b^2}{3l^4} - \frac{p^2}{280}\right) - \left(\frac{2}{5}\frac{Eg}{\gamma}\frac{b^2}{3l^4} - \frac{p^2}{105}\right)^2 = 0. \qquad (m)$$

From this equation p^2 can be calculated. The smallest of the two roots gives

$$f = \frac{p}{2\pi} = \frac{5.319}{2\pi}\frac{b}{l^2}\sqrt{\frac{Eg}{3\gamma}}. \qquad (n)$$

It is interesting to note that for the case under consideration an exact solution exists in which the forms of the normal functions are determined in terms of Bessel's functions.* From this exact solution we have

$$f = \frac{p}{2\pi} = \frac{5.315}{2\pi}\frac{b}{l^2}\sqrt{\frac{Eg}{3\gamma}}. \qquad (168)$$

Comparing with (l) and (n) it can be concluded that the accuracy of the first approximation is about 3%, while the error of the second approximation is less than .1% and a further increase in the number of terms in expression (e) is necessary only if the frequencies of the higher modes of vibration are also to be calculated.

* See G. Kirchhoff, Berlin, Monatsberichte, p. 815 (1879), or Ges. Abhandlungen, p. 339. See also Todhunter and Pearson, A History of the Theory of Elasticity, Vol. 2, part 2, p. 92.

For comparison it is important to note that in the case of a prismatical cantilever bar having the same section as the wedge at the thick end, the following result was obtained (see p. 234)

$$f = \frac{p}{2\pi} = \frac{a1.875^2}{2\pi l^2} = \frac{3.515b}{2\pi l^2}\sqrt{\frac{Eg}{3\gamma}}.$$

The method developed above can be applied also in cases when A and I are not represented by continuous functions of x. These functions may have several points of discontinuity or may be represented by different mathematical expressions in different intervals along the length l. In such cases the integrals (h) should be subdivided into intervals such that I and A may be represented by continuous functions in each of these intervals. If the functions A and I are obtained either graphically or from numerical tables this method can also be used, it being only necessary to apply one of the approximate methods in calculating the integrals (h). This makes Ritz's method especially suitable in studying the vibration of turbine blades and such structures as bridges and hulls of ships.

Vibration of a Conical Rod.—The problem of the vibrations of a conical rod which has its tip free and the base built in was first treated by Kirchhoff.* For the fundamental mode he obtained in this case

$$f = \frac{p}{2\pi} = \frac{4.359}{2\pi}\frac{r}{l^2}\sqrt{\frac{Eg}{\gamma}}, \tag{169}$$

where r = radius of the base,

l = the length of the rod.

For comparison it should be remembered here that a cylindrical rod of the same length and area of base has the frequency (see p. 234)

$$f = \frac{p}{2\pi} = \frac{a}{2\pi}\frac{1.875^2}{l^2} = \frac{1.758}{2\pi}\frac{r}{l^2}\sqrt{\frac{Eg}{\gamma}}.$$

Thus the frequencies of the fundamental modes of a conical and a cylindrical rod are in the ratio 4.359 : 1.758. The frequencies of the higher modes of vibration of a conical rod can be calculated from the equation

$$f = \frac{p}{2\pi} = \frac{\alpha}{2\pi}\frac{r}{l^2}\sqrt{\frac{Eg}{\gamma}}, \tag{170}$$

* Loc. cit., p. 267.

in which α has the values given below.*

α_1	α_2	α_3	α_4	α_5	α_6
4.359	10.573	19.225	30.339	43.921	59.956

Other Cases of Vibration of a Cantilever of Variable Cross Section.—In the general case the frequency of the lateral vibrations of a cantilever can be represented by the equation

$$f = \frac{p}{2\pi} = \frac{\alpha}{2\pi}\frac{i}{l^2}\sqrt{\frac{Eg}{\gamma}}, \tag{171}$$

in which i = radius of gyration of the built-in section,

l = length of the cantilever,

α = constant depending on the shape of the bar and on the mode of vibration.

In the following the values of this constant α for certain particular cases of practical importance are given.

1. If the variations of the cross sectional area and of the moment of inertia, along the axis x, can be expressed in the form,

$$A = ax^m; \qquad I = bx^m, \tag{172}$$

x being measured from the free end, i remains constant along the length of the cantilever and the constant α, in eq. (171) can be represented for the fundamental mode with sufficient accuracy by the equation †

$$\alpha = 3.47(1 + 1.05m).$$

2. If the variation of the cross sectional area and of the moment of inertia along the axis x can be expressed in the form

$$A = a\left(1 - c\frac{x}{l}\right); \qquad I = b\left(1 - c\frac{x}{l}\right), \tag{173}$$

x being measured from the built-in end, then i remains constant along the length of the rod and the quantity α, in eq. (171), will be as given in the table below.‡

$c =$	0	.4	.6	.8	1.0
$\alpha =$	3.515	4.098	4.585	5.398	7.16

* See, Dorothy Wrinch, Proc. Roy. Soc. London, Vol. 101 (1922), p. 493.

† See Akimasa Ono, Journal of the Society of Mechanical Engineers, Tokyo, Vol. 27 (1924), p. 467.

‡ Akimasa Ono, Journal of the Society of Mechanical Engineers, Vol. 28 (1925), p. 429.

Bar of Variable Cross Section with Free Ends.—Let us consider now the case of a laterally vibrating free-free bar consisting of two equal halves joined together at their thick ends (fig. 126), the left half being generated by revolving the curve

$$y = ax^n \tag{o}$$

about the x axis. The exact solution in terms of Bessel functions has been obtained in this case for certain values of n * and the frequency of the fundamental mode can be represented in the form

$$f = \frac{p}{2\pi} = \frac{\alpha r}{4\pi l^2}\sqrt{\frac{Eg}{\gamma}}, \tag{174}$$

in which r = radius of the thickest cross section,

$2l$ = length of the bar,

α = constant, depending on the shape of the curve (*o*), the values of which are given in the table below:

$n =$	0	1/4	1/2	3/4	1
$\alpha =$	5.593	6.957	8.203	9.300	10.173

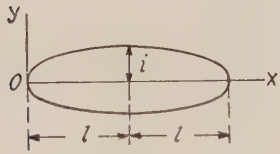


Fig. 126.

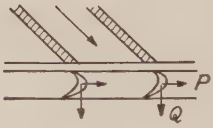


Fig. 127.

The application of integral equations in investigating lateral vibrations of bars of variable cross section has been discussed by E. Schwerin.†

50. Vibration of Turbine Blades.—*General.*—It is well known that under certain conditions dangerous vibrations in turbine blades may occur and to this fact the majority of fractures in such blades may be attributed. The disturbing force producing the vibrations in this case is the steam pressure. This pressure always can be resolved into two components; a tangential component P and an axial one Q (Fig. 127) which produce bending of blades in the tangential and axial directions, respectively. These components do not remain constant, but vary with the time because they depend on the relative position of the moving blades with

* See J. W. Nicholson; Proc. Roy. Soc. of London, Vol. 93 (1917), p. 506.

† E. Schwerin, Über Transversalschwingungen von Stäben veraenderlichen Querschnitts. Zeitschr. f. techn. Physik, Vol. 8, 1927, p. 264.

respect to the fixed guide blades. Such pulsating forces, if in resonance with one of the natural modes of vibration of the blades, may produce large forced vibrations with consequent high stresses, which may result finally in the production of progressive fatigue cracks at points of sharp variation in cross section, where high stress concentration takes place. From this it can be seen that the study of vibration of turbine blades and the determination of the various frequencies corresponding to the natural modes of vibration may assist the designer in choosing such proportions for the blades that the possibility of resonance will be eliminated. In making such investigations, Rayleigh's method usually gives a satisfactory approximation. It is therefore unnecessary to go further in the refinement of the calculations, especially if we take into consideration that in actual cases variations in the condition at the built-in end of the blade may affect considerably the frequencies of the natural modes of vibration.*

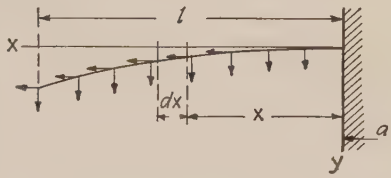


FIG. 128.

Due to the fact that the two principal moments of inertia of a cross section of a blade are different, natural modes of vibration in two principal planes should be studied separately.

Application of Rayleigh's Method.—Let xy be one of these two principal planes (Fig. 128).

l = length of the blade.

a = the radius of the rotor at the built-in end of the blade.

c = constant defined by eq. (173).

A = cross sectional area of the blade varying along the x axis.

ω = angular velocity of the turbine rotor.

γ = weight of material per unit volume.

X = function of x representing the deflection curve of the blade under the action of its weight.

Taking the curve represented by the function X as a basis for the calculation of the fundamental mode of vibration, the deflection curve of the blade during vibration will be,

$$y = X \cos pt. \tag{a}$$

* See, W. Hort, V. D. I., Vol. 70 (1926), p. 1420.

The maximum potential energy will be obtained when the blade is in its extreme position and the deflection curve is represented by the equation

$$y = X. \tag{b}$$

This energy consists of two parts: (1) the energy V_1 due to lateral bending and (2) the energy V_2 due to the action of the centrifugal forces. The energy V_1 is equal to the work done by the lateral loading during the deflection, given by eq. (*b*), and is represented by the equation:

$$V_1 = \frac{\gamma}{2}\int_0^l AXdx, \tag{c}$$

in which X, the function of x representing the deflection curve of the blade produced by its weight, can always be obtained by analytical or graphical methods. In the latter case the integral (*c*) can be calculated by one of the approximate methods.

In calculating V_2 it should be noted that the centrifugal force acting on an element of the length dx of the blade (see Fig. 128) is

$$\frac{A\gamma dx}{g}\omega^2(a + x). \tag{d}$$

The radial displacement of this element towards the center due to bending of the blade is

$$\frac{1}{2}\int_0^x \left(\frac{dX}{dx}\right)^2 dx, \tag{e}$$

and the work of the centrifugal force (*d*) will be

$$-\frac{A\gamma}{2g}dx\omega^2(a + x)\int_0^x \left(\frac{dX}{dx}\right)^2 dx. \tag{f}$$

The potential energy V_2 will now be obtained by the summation of the elements of work (*f*), along the length of the blade and by changing the sign of the sum. Then

$$V_2 = \frac{\gamma\omega^2}{2g}\int_0^l A(a + x)dx\int_0^x \left(\frac{dX}{dx}\right)^2 dx. \tag{g}$$

The maximum kinetic energy will be obtained when the vibrating blade is in its middle position and the velocities, calculated from equation (*a*) have the values:

$$\dot{y} = pX.$$

Then

$$T = \frac{1}{2}\int_0^l \frac{A\gamma}{g}\dot{y}^2 dx = \frac{\gamma p^2}{2g}\int_0^l AX^2 dx. \tag{h}$$

Now, from the equation

$$T = V_1 + V_2$$

we obtain

$$p^2 = \frac{g\int_0^l AXdx + \omega^2\int_0^l A(a+x)dx\int_0^x\left(\frac{dX}{dx}\right)^2 dx}{\int_0^l AX^2 dx}. \tag{175}$$

This is the equation for calculating the frequencies corresponding to the natural modes of vibration of a blade.

The second term in the numerator of the right hand member represents the effect of centrifugal force. Denoting by

$$f_1^2 = \frac{g}{(2\pi)^2}\frac{\int_0^l AXdx}{\int_0^l AX^2 dx}; \quad f_2^2 = \frac{\omega^2}{(2\pi)^2}\frac{\int_0^l A(a+x)dx\int_0^x\left(\frac{dX}{dx}\right)^2 dx}{\int_0^l AX^2 dx}. \tag{176}$$

we find, from eq. (175), that the frequency of vibration of the blade can be represented in the following form:

$$f = \sqrt{f_1^2 + f_2^2}, \tag{177}$$

in which f_1 denotes the frequency of the blade when the rotor is stationary, and f_2 represents the frequency of the blade when the elastic forces are neglected and only the restitutive force due to centrifugal action is taken into consideration.

Vibration in the Axial Direction.—In calculating the frequency of vibration in an axial direction a good approximation can be obtained by assuming that the variation of the cross sectional area and of the moment of inertia along the axis of the blade is given by the equations (173). In this case the frequency f_1 will be obtained by using the corresponding table (see p. 269).

The frequency f_2, for the same case, can be easily calculated from eq. (176) and can be represented in the following form

$$f_2 = \frac{\beta\omega}{2\pi}, \tag{178}$$

in which β is a number depending on the proportions of the blade.

Several values of β are given in the table below.* Knowing f_1 and f_2 the frequency f will now be obtained from eq. (177).

a/l = \ c =	0	.2	.4	.6	.8	1.0
0	1.00	1.00	1.00	1.00	1.00	1.00
1	1.57	1.58	1.59	1.61	1.64	1.71
2	1.98	2.00	2.01	2.04	2.09	2.19
4	2.62	2.64	2.66	2.70	2.77	2.92
6	3.13	3.15	3.18	3.23	3.31	3.50
8	3.56	3.59	3.62	3.68	3.78	4.00
10	3.95	3.98	4.02	4.08	4.19	4.44

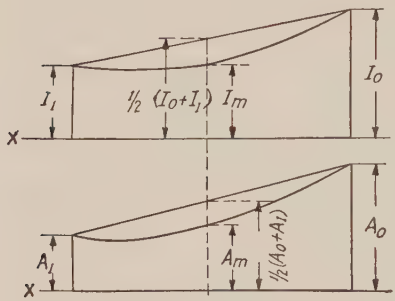


FIG. 129.

Vibration in the Tangential Direction.—In the tangential direction the blades have usually a variable radius of gyration. Consequently the equations (173) cannot be directly applied. In such a case an approximation can be obtained by assuming that the variation of I and A along the x axis (Fig. 129) can be represented by the equations:†

$$I = I_0\left(1 - m\frac{x}{l} - m' \sin\frac{\pi x}{l}\right),$$
$$A = A_0\left(1 - n\frac{x}{l} - n' \sin\frac{\pi x}{l}\right), \tag{179}$$

in which

$$m = \frac{I_0 - I_1}{I_0}; \qquad n = \frac{A_0 - A_1}{A_0};$$
$$m' = \frac{1}{I_0}\left(\frac{I_0 + I_1}{2} - I_m\right); \qquad n' = \frac{1}{A_0}\left(\frac{A_0 + A_1}{2} - A_m\right).$$

The frequencies will then be calculated from the general eq. (171) in which the constant α for the fundamental and higher modes of vibration is given by the equation ‡

* The table is taken from the paper by Akimasa Ono, mentioned above, p. 269.

† W. Hort: Proceedings of the First International Congress for Applied Mechanics, Delft, (1925) p. 282. The numerical results, given below, are obtained on the assumption that the mode of vibration of a bar of variable cross section is the same as that of a prismatical bar.

‡ If the values of m, m', n, and n' are not greater than .5, formula (180) according to Hort, is correct to within 2%. To get an idea of the error made in case m and n were

$$\alpha_i = \alpha_{0i}\sqrt{\frac{1 - m\beta_i - m'\beta_i'}{1 - n\gamma_i - n'\gamma_i'}}. \tag{180}$$

Here, α_{0i} = values of the constant α for a cantilever of uniform section (see table on p. 234).* The constants β_i, β_i', γ_i and γ_i' for the various modes of vibration are given in the table below.

i	β_i	γ_i	β_i'	γ_i'
1	.193	.807	.493	.493
2	.405	.594	.703	.703
3	.468	.532	.661	.661
4	.483	.517	.649	.649
5	.490	.510	.645	.645
6	.493	.507	.642	.642

If one end of the blade is built in while the other is simply supported, the same equation (180) can be used in calculating α_i. In this case α_{0i} should be taken from the second table on p. 234. The constants β_i, β_i', γ_i and γ_i' are given in the table below.

i	β_i	γ_i	β_i'	γ_i'
1	.431	.569	.626	.857
2	.480	.520	.612	.724
3	.490	.510	.623	.680
4	.494	.506	.628	.662
5	.496	.504	.631	.654
6	.497	.503	.633	.649

In this manner f_1 in eq. (177) can be calculated. For calculating f_2 for the fundamental mode, eq. (178) and the above table can be used and the frequency f will then be obtained from eq. (177) as before.

It should be noted that the blades are usually connected in groups by means of shrouding wires. These wires do not always substantially affect the frequencies of the axial vibrations but they may change the frequencies of the tangential vibrations considerably.†

unity, the exact solutions for the natural frequencies of a conical shaped blade and a wedge shaped blade were compared with the values obtained by the above method. It was found that in these extreme cases the error was 17% and 18.5%, respectively, for the conical shaped blade and the wedge shaped blade.

* $k_i^2l^2$ of this table is equal to α_{0i} in eq. (180).

† See Stodola's book, loc. cit. p. 949. See also W. Hort, V. D. I. Vol. 70 (1926) p. 1422 and E. Schwerin, Über die Eigenfrequenzen der Schaufelgruppen von Dampfturbinen, Zeitschr. f. techn. Physik, Vol. 8, 1927, p. 312.

51. Vibration of Hulls of Ships.—As another example of the application of the theory of vibration of bars of variable section, the problem of the vibration of the hull of a ship will now be considered. The disturbing force in this case is usually due to unbalance in the engine and, if the frequency of this force coincides with the frequency of one of the natural modes of vibration of the hull, large forced vibrations may be produced. If the hull of the ship be taken as a bar of variable section with free ends and Ritz's method (see paragraph 48) be applied, the frequencies of the various modes can always be calculated with sufficient accuracy from the equations (167).

To simplify the problem let us assume that the bar is symmetrical with respect to the middle cross section and that, by putting the origin of coordinates in this section, the cross sectional area and moment of inertia for any cross section can be represented, respectively, by the equations

$$A = A_0(1 - cx^2); \qquad I = I_0(1 - bx^2), \tag{a}$$

in which A_0 and I_0 denote the cross sectional area and the moment of inertia of the middle cross section, respectively. It is understood that x may vary from $x = -l$ to $x = +l$, $2l$ being the length of the ship.

We will further assume that the deflection during vibration may be represented by

$$y = X \cos pt,$$

in which X is taken in the form of the series,

$$X = a_1\varphi_1(x) + a_2\varphi_2(x) + a_3\varphi_3(x) + \cdots. \tag{b}$$

We must choose for $\varphi_1, \varphi_2, \cdots$ suitable functions, satisfying the end conditions. The ratios between the coefficients $a_1, a_2, a_3, \cdots$ and the frequencies will be then obtained from the equations (167).

A satisfactory approximation for the frequency of the fundamental mode of vibration can be obtained * by taking for the functions $\varphi(x)$ the normal functions for a prismatical bar with free ends. The general solution (133) for symmetrical modes of vibration should be taken in the form

$$X = C_1(\cos kx + \cosh kx) + C_2(\cos kx - \cosh kx). \tag{c}$$

Now from the conditions at the free ends we have

$$(X'')_{x=\pm l} = (X''')_{x=\pm l} = 0. \tag{d}$$

* See author's book, "Theory of Elasticity," Vol. 2 (1916), S. Petersburg. See also N. Akimoff, Trans. of the Soc. of Naval Arch. (New York), Vol. 26 (1918).

Substituting (c) in (d), we obtain

$$C_1(-\cos kl + \cosh kl) - C_2(\cos kl + \cosh kl) = 0, \qquad (e)$$
$$C_1(\sin kl + \sinh kl) - C_2(-\sin kl + \sinh kl) = 0.$$

Putting the determinant of these equations equal to zero the frequency equation

$$\tan kl + \tanh kl = 0, \qquad (f)$$

will be obtained, the consecutive roots of which are

$$k_1 l = 0; \qquad k_2 l = 2.3650 \cdots.$$

Substituting from (e) the ratio C_1/C_2 into eq. (c) the normal functions corresponding to the fundamental and higher modes of vibration will be

$$X_i = C_i(\cos k_i x \cosh k_i l + \cosh k_i x \cos k_i l).$$

The arbitrary constant, for simplification, will be taken in the form

$$C_i = \frac{1}{\sqrt{\cos^2 k_i l + \cosh^2 k_i l}}.$$

The normal function, corresponding to the first root, $k_1 l = 0$, will be a constant and the corresponding motion will be a displacement of the bar as a rigid body in the y direction. This constant will be taken equal to $1/\sqrt{2}$.

Taking the normal functions, obtained in this manner, as suitable functions $\varphi(x)$ in the series (b) we obtain

$$X = a_1 \frac{1}{\sqrt{2}} + a_2 \frac{\cos k_2 x \cosh k_2 l + \cosh k_2 x \cos k_2 l}{\sqrt{\cos^2 k_2 l + \cosh^2 k_2 l}} + \cdots. \qquad (g)$$

Substituting the above in eq. (167), we obtain

$$\frac{\partial}{\partial a_n}\left\{ I_0 \int_{-l}^{+l} (1 - bx^2) \sum_{i=1,2,3} \sum_{j=1,2,3} a_i a_j \varphi_i'' \varphi_j'' dx \right.$$
$$\left. - \frac{p^2 A_0 \gamma}{Eg} \int_{-l}^{+l} (1 - cx^2) \sum_{i=1,2,3,\cdots} \sum_{j=1,2,3,\cdots} a_i a_j \varphi_i \varphi_j dx \right\} = 0 \qquad (h)$$

and denoting

$$\int_{-l}^{+l} (1 - bx^2) \varphi_i'' \varphi_j'' dx = \alpha_{ij}; \qquad \int_{-l}^{+l} (1 - cx^2) \varphi_i \varphi_j dx = \beta_{ij} \qquad (k)$$

we obtain, from (h),

$$\sum_{i=1,2,3,\cdots} a_i(\alpha_{in} - \lambda\beta_{in}) = 0, \tag{l}$$

in which

$$\lambda = \frac{p^2 A_0 \gamma}{E I_0 g}. \tag{m}$$

For determining the fundamental mode of vibration two terms of the series (g) are practically sufficient. The equations (l) in this case become

$$\begin{aligned} a_1(\alpha_{11} - \lambda\beta_{11}) + a_2(\alpha_{21} - \lambda\beta_{21}) &= 0, \\ a_1(\alpha_{12} - \lambda\beta_{12}) + a_2(\alpha_{22} - \lambda\beta_{22}) &= 0. \end{aligned} \tag{n}$$

In our case,

$$\varphi_1'' = 0; \qquad \varphi_2'' = k_2^2 \frac{-\cos k_2 x \cosh k_2 l + \cosh k_2 x \cos k_2 l}{\sqrt{\cos^2 k_2 l + \cosh^2 k_2 l}}.$$

Substituting this in (k) and performing the integration, we obtain

$$\alpha_{11} = 0; \qquad \alpha_{12} = 0; \qquad \alpha_{21} = 0,$$

$$\alpha_{22} = \int_{-l}^{+l} (1 - bx^2)(\varphi_2'')^2 dx = \frac{31.28}{l^3}(1 - .087bl^2), \tag{p}$$

$$\beta_{11} = l(1 - .333cl^2); \qquad \beta_{12} = \beta_{21} = .297cl^3; \qquad \beta_{22} = l(1 - .481cl^2). \tag{q}$$

Substituting in eqs. (n) and equating the determinant of these equations to zero, the frequency equation becomes:

$$\lambda^2\left(1 - \frac{\beta_{12}^2}{\beta_{11}\beta_{22}}\right) - \lambda\frac{\alpha_{22}}{\beta_{22}} = 0. \tag{r}$$

The first root of this equation ($\lambda = 0$) corresponds to a displacement of the bar as a rigid body. The second root

$$\lambda = \frac{\alpha_{22}}{\beta_{22}} \frac{1}{1 - \dfrac{\beta_{12}^2}{\beta_{11}\beta_{22}}} \tag{s}$$

determines the frequency of the fundamental type of vibration. This frequency is

$$f_1 = \frac{p}{2\pi} = \frac{\sqrt{\lambda}}{2\pi}\sqrt{\frac{E I_0 g}{A_0 \gamma}}. \tag{t}$$

Numerical Example.—Let $2l = 100$ meters; $J_0 = 20$ (meter)^4;

$A_0\gamma = 7 \times 9.81$ ton per meter; $b = c = .0003$ per meter square. Then the weight of the ship

$$Q = 2A_0\gamma \int_0^l (1 - cx^2)dx = 5150 \text{ ton.}$$

From eqs. (p) and (q) we obtain

$$\alpha_{22} = 23.40 \times 10^{-5}; \qquad \beta_{11} = 37.50; \qquad \beta_{12} = 11.14; \qquad \beta_{22} = 31.95;$$

then, from eq. (s), we get

$$\lambda = .817 \times 10^{-5}.$$

Assuming $E = 2.10^7$ ton per meter square, we obtain

$$p = \sqrt{\frac{20}{7} \times 2 \times 10^7 \times .817 \times 10^{-5}} = 21.6.$$

The number of oscillations per minute

$$N = \frac{60p}{2\pi} = 206.$$

The functions $\varphi(x)$, taken above, can be used also when the laws of variation of I and A are different from those given by eqs. (a) and also when I and A are given graphically. In each case it is only necessary to calculate the integrals (k) which calculation can always be carried out by means of some approximate method.

52. Lateral Impact of Bars.—*Approximate Solution.*—The problem of stresses and deflections produced in a beam by a falling body is of great practical importance. The exact solution of this problem involves the study of the lateral vibration of the beam. In cases where the mass of the beam is negligible in comparison with the mass of the falling body an approximate solution can easily be obtained by assuming that the deflection curve of the beam during impact has the same shape as the corresponding statical deflection curve. Then the maximum deflection and the maximum stress will be found from a consideration of the energy of the system. Let us take, for example, a beam supported at the ends and struck midway between the supports by a falling weight W. If δ denotes the deflection at the middle of the beam the following relation between the deflection and the force P acting on the beam holds:

$$\delta = \frac{Pl^3}{48EI}$$

and the potential energy of deformation will be

$$V = \frac{P\delta}{2} = \frac{24EI\delta^2}{l^3}. \tag{a}$$

If the weight W falls through a height h, the work done by this load during falling will be

$$W(h + \delta_d) \tag{b}$$

and the dynamical deflection δ_d will be found from the equation,

$$W(h + \delta_d) = \frac{24EI\delta_d^2}{l^3}, \tag{c}$$

from which

$$\delta_d = \delta_{st} + \sqrt{\delta_{st}^2 + 2h\delta_{st}}, \tag{d}$$

where

$$\delta_{st} = \frac{Wl^3}{48EI}$$

represents the statical deflection of the beam under the action of the load W.

In the above discussion the mass of the beam was neglected and it was assumed that the kinetic energy of the falling weight W was completely transformed into potential energy of deformation of the beam. In actual conditions a part of the kinetic energy will be lost during the impact. Consequently calculations made as above will give an upper limit for the dynamical deflection and the dynamical stresses. In order to obtain a more accurate solution the mass of a beam subjected to impact must be taken into consideration.

If a moving body, having a mass W/g and a velocity v_0 strikes centrally a stationary body of mass w/g, and, if the deformation at the point of contact is perfectly inelastic, the final velocity v, after the impact (equal for both bodies), may be determined from the equation

$$\frac{W}{g}v_0 = \frac{W + w}{g}v,$$

from which

$$v = v_0\frac{W}{W + w}. \tag{e}$$

It should be noted that for a beam at the instance of impact, it is only at the point of contact that the velocity v of the body W and of the beam will be the same. Other points of the beam may have velocities different from v. For instance, at the supports of the beam these

velocities will be equal to zero. Therefore, not the actual mass of the beam, but some *reduced mass* must be used in eq. (*e*) for calculating the velocity v. The magnitude of this reduced mass will depend on the shape of the deflection curve and can be approximately determined in the same manner as was done in Rayleigh's method (see eq. 42, p. 56), i.e., by assuming that the deflection curve is the same as the one obtained statically. Then

$$v = v_0 \frac{W}{W + \frac{17}{35} w},$$

in which $17/35w$ is the *reduced weight* of the beam. The kinetic energy of the system will be

$$\frac{\left(W + \frac{17}{35} w\right) v^2}{2g} = \frac{W v_0^2}{2g} \frac{1}{1 + \frac{17}{35} \frac{w}{W}}.$$

This quantity should be substituted for $(Wv_0^2/2g) = Wh$ in the previous equation (*c*) in order to take into account the effect of the mass of the beam. The dynamical deflection then becomes

$$\delta_d = \delta_{st} + \sqrt{\delta_{st}^2 + 2h\delta_{st} \frac{1}{1 + \frac{17}{35} \frac{w}{W}}}. \tag{181}$$

The same method can be used in all other cases of impact in which the displacement of the structure at the point of impact is proportional to the force.*

Impact and Vibrations.—The method described above gives sufficiently accurate results for the cases of thin rods and beams if the mass of the falling weight is large in comparison to the mass of the beam. Otherwise the consideration of vibrations of the beam and of local deformations at the point of impact becomes necessary.

Lateral vibrations of a beam struck by a body moving with a given velocity were considered by S. Venant.† Assuming that after impact the striking body becomes attached to the beam, the vibrations can be investigated by expressing the deflection as the sum of a series of

* This method was developed by H. Cox, Cambridge Phil. Soc. Trans., Vol. 9 (1850), p. 73. See also Todhunter and Pearson, History, Vol. 1, p. 895.

† Loc. cit., p. 247, note finale du paragraphe 61, p. 490.

normal functions. The constant coefficients of this series should be determined in such a manner as to satisfy the given initial conditions. In this manner, S. Venant was able to show that the approximate solution given above has an accuracy sufficient for practical applications.

The assumption that after impact the striking body becomes attached to the beam is an arbitrary one and in order to get a more accurate picture of the phenomena of impact, the local deformations of the beam and of the striking body at the point of contact should be investigated. Some results of such an investigation in which a ball strikes the flat surface of a rectangular beam will now be given.* The local deformation will be given in this case by the known solution of Herz.† Let α denote the displacement of the striking ball with respect to the axis of the beam due to this deformation and P, the corresponding pressure of the ball on the beam; then

$$\alpha = kP^{2/3}, \tag{f}$$

where k is a constant depending on the elastic properties of the bodies and on the magnitude of the radius of the ball. The pressure P, during impact, will vary with the time and will produce a deflection of the beam which can be expressed by the general solution (c) of paragraph 44. If the beam is struck at the middle, the expression for the generalized forces will be

$$Q_i = P \sin \frac{i\pi}{2}$$

and the deflection at the middle produced by the pressure P becomes

$$y = \sum_{i=1,3,5,\cdots}^{i=\infty} \frac{1}{i^2} \frac{l^2}{\pi^2 a} \frac{2g}{\gamma A l} \int_0^t P \sin \frac{i^2\pi^2 a(t - t_1)dt_1}{l^2}. \tag{g}$$

The complete displacement of the ball from the beginning of the impact ($t = 0$) will be equal to

$$d = \alpha + y. \tag{h}$$

The same displacement can be found now from a consideration of the motion of the ball. If v_0 is the velocity of the ball at the beginning of the impact ($t = 0$) the velocity v at any moment $t = t_1$ will be equal to‡

$$v = v_0 - \frac{1}{m} \int_0^{t_1} P dt_1, \tag{k}$$

* See author's paper, Zeitschr. f. Math. u. Phys., Vol. 62 (1913), p. 198.

† H. Herz: J. f. Math. (Crelle), Vol. 92 (1881). A. E. H. Love, Math. Theory of Elasticity (1927), p. 198.

‡ It is assumed that no forces other than P are acting on the ball.

in which m = the mass of the ball and P = the reaction of the beam on the ball varying with the time. The displacement of the ball in the direction of impact will be,

$$d = v_0 t - \int_0^t \frac{dt_1}{m} \int_0^{t_1} P dt_1. \qquad (l)$$

Equating (h) and (l) the following equation is obtained,

$$v_0 t - \int_0^t \frac{dt_1}{m} \int_0^{t_1} P dt_1 = kP^{2/3}$$
$$+ \sum_{i=1,3,5,\cdots}^{i=\infty} \frac{1}{i^2} \frac{l^2}{\pi^2 a} \frac{2g}{\gamma A l} \int_0^t P \sin \frac{i^2 \pi^2 a (t - t_1) dt_1}{l^2}. \qquad (m)$$

This equation can be solved numerically by sub-dividing the interval of time from 0 to t into small elements and calculating, step by step, the displacements of the ball. In the following the results of such calculations for two numerical examples are given.

Examples.—In the first example a steel bar of a square cross section 1×1 cm. and of length = 15.35 cm. is taken. A steel ball of the radius $r = 1$ cm. strikes the bar with a velocity $v = 1$ cm. per sec. Assuming $E = 2.2 \times 10^6$ kilograms per sq. cm. and $\gamma = 7.96$ grams per cu. cm. the period of the fundamental mode of vibration will be $\tau = .001$ sec. In the numerical solution of eq. (m) this period was sub-divided into 180 equal parts so that $\delta\tau = (1/180)\tau$. The pressure P calculated for each step is given in Fig. 130 by the curve I. For comparison in the same figure the variation of pressure with time, for the case when the ball strikes an infinitely large body having a plane boundary surface is shown by the dotted lines. It is seen that the ball remains in contact with the bar only during an interval of time equal to 28 ($\delta\tau$), i.e., about 1/6 of τ. The displacements of the ball are represented by curve II and the deflection of the bar at the middle by curve III.

A more complicated case is represented in Fig. 131. In this case the length of the bar and the radius of the ball are taken twice as great as in the previous example. The period τ of the fundamental mode of vibration of the bar is four times as large as in the previous case while the variation of the pressure P is represented by a more complicated curve I. It is seen that the ball remains in contact with the bar from $t = 0$ to $t = 19.5(\delta\tau)$. Then it strikes the bar again at the moment $t = 60(\delta\tau)$ and remains in contact till $t = 80(\delta\tau)$. The increase in deflection of the bar is given by curve II.

It will be noted from these examples that the phenomenon of elastic

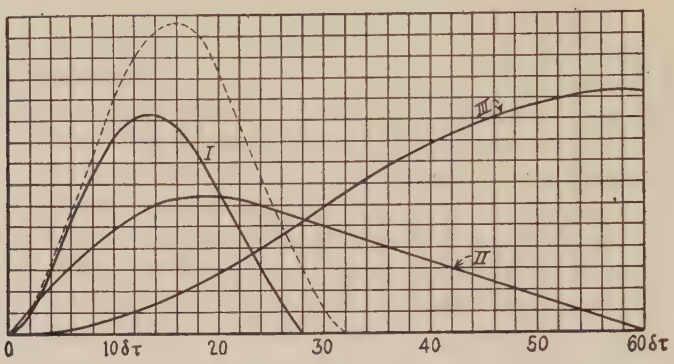


FIG. 130.

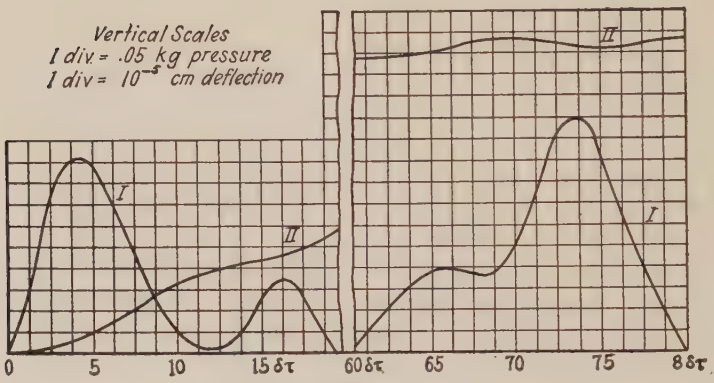


FIG. 131.

impact is much more complicated than that of inelastic impact considered by S. Venant.

53. Longitudinal Impact of Prismatical Bars.—*General.*—For the approximate calculation of the stresses and deflections produced in a prismatical bar, struck longitudinally by a moving body, the approximate method developed in the previous paragraph can be used, but for a more accurate solution of the problem a consideration of the longitudinal vibrations of the bar is necessary.

Young was the first * to point out the necessity of a more detailed

* See his Lectures on Natural Philosophy, Vol. I, p. 144. The history of the longitudinal impact problem is discussed in detail in the book of Clebsch, translated by S. Venant, loc. cit. p. 247 see *note finale du par.* 60, p. 480, *a.*

consideration of the effect of the mass of the bar on the longitudinal impact. He showed also that any small perfectly rigid body will produce a permanent set in the bar during impact, provided the ratio of the velocity v_1 of motion of the striking body to the velocity v of the propagation of sound waves in the bar is larger than the strain corresponding to the elastic limit in compression of the material. In order to prove this statement he assumed that at the moment of impact (Fig. 132) a local compression will be produced * at the surface of contact of the moving body and the bar which compression is propagated along the bar with the velocity of sound. Let us take a very small interval of time equal to t, such that during this interval the velocity of the striking body can be considered as unchanged. Then the displacement of the body will be v_1t and the length of the compressed portion of the bar will be vt. Consequently the unit compression becomes equal to v_1/v. (Hence the statement mentioned above.)

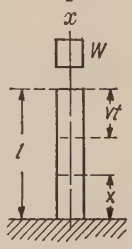


FIG. 132

The longitudinal vibrations of a prismatical bar during impact were considered by Navier.† He based his analysis on the assumption that after impact the moving body becomes attached to the bar at least during a half period of the fundamental type of vibration. In this manner the problem of impact becomes equivalent to that of the vibrations of a load attached to a prismatical bar and having at the initial moment a given velocity (see paragraph 38). The solution of this problem, in the form of an infinite series given before, is not suitable for the calculation of the maximum stresses during impact and in the following a more comprehensive solution, developed by S. Venant ‡ and J. Boussinesq,§ will be discussed.

Bar fixed at One End and Struck at the Other.||—Considering first the bar fixed at one end and struck longitudinally at the other, Fig. 132, recourse will be taken to the already known equation for longitudinal vibrations (see p. 200). This equation is

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \qquad (a)$$

in which u denotes the longitudinal displacements from the position of

* It is assumed that the surfaces of contact are two parallel smooth planes.

† Rapport et Mémoire sur les Ponts Suspendus, Ed. (1823).

‡ Loc. cit., p. 247.

§ Applications des Potentiels, p. 508.

|| See Love, "Theory of Elasticity," 4th ed., p. 431 (1927).

equilibrium during vibration and

$$a^2 = \frac{Eg}{\gamma}. \tag{b}$$

The condition at the fixed end is

$$(u)_{x=0} = 0. \tag{c}$$

The condition at the free end, at which the force in the bar must be equal to the inertia force of the striking body, will be

$$AE\left(\frac{\partial u}{\partial x}\right)_{x=l} = -\frac{W}{g}\left(\frac{\partial^2 u}{\partial t^2}\right)_{x=l}. \tag{d}$$

Denoting by m the ratio of the weight W of the striking body to the weight $A\gamma l$ of the bar, we obtain, from (d)

$$ml\left(\frac{\partial^2 u}{\partial t^2}\right)_{x=l} = -a^2\left(\frac{\partial u}{\partial x}\right)_{x=l} \tag{e}$$

The conditions at the initial moment $t = 0$, when the body strikes the bar, are

$$u = \frac{\partial u}{\partial t} = 0 \tag{f}$$

for all values of x between $x = 0$ and $x = l$ while at the end $x = l$, since at the instant of impact the velocity of the struck end of the bar becomes equal to that of the striking body, we have:

$$\left(\frac{\partial u}{\partial t}\right)_{t=+0} = -v. \tag{g}$$

The problem consists now in finding such a solution of the equation (a) which satisfies the terminal conditions (c) and (e) and the initial conditions (f) and (g).

The general solution of this equation can be taken in the form

$$u = f(at - x) + f_1(at + x), \tag{h}$$

in which f and f_1 are arbitrary functions.

If accents indicate differentiation with respect to the arguments $(at - x)$ or $(at + x)$ we may put

$$\frac{\partial f}{\partial x} = -\frac{\partial f_1}{\partial x} = -f'(at - x); \qquad \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f_1}{\partial x^2} = f''(at - x),$$

$$\frac{\partial f}{\partial t} = \frac{\partial f_1}{\partial t} = af'(at - x); \qquad \frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f_1}{\partial t^2} = a^2 f''(at - x);$$

from which it is seen that the expression (h) satisfies eq. (a).

In order to satisfy the terminal condition (c) we must have,

$$f(at) + f_1(at) = 0$$

or

$$f_1(at) = -f(at)$$

for any value of the argument at. Hence the solution (h) may be written in the form

$$u = f(at - x) - f(at + x). \tag{k}$$

This solution has a very simple physical meaning which can be easily explained in the following manner. Let us take the first term $f(at - x)$ on the right side of eq. (k) and consider a certain instant t. The function f can be represented for this instant by some curve nsr (Fig. 133), the shape of which will depend on the function f. It is easy to see that after the lapse of an element of time Δt the argument $at - x$ of the function f will remain unchanged provided only that the abscissae are increased during the same interval of time by an element Δx equal to $a\Delta t$. Geometrically this means that during the interval of time Δt the curve nsr moves without distortion to a new position shown in the figure by the dotted line. It can be appreciated from this consideration that the first term on the right side of eq. (k) represents a wave traveling along the x axis with a constant velocity equal to

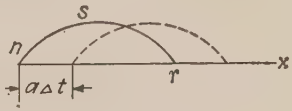


Fig. 133.

$$a = \sqrt{\frac{Eg}{\gamma}}, \tag{182}$$

which is also the velocity of propagation of sound waves along the bar. In the same manner it can be shown that the second term on the right side of eq. (k) represents a wave traveling with the velocity a in the negative direction along the x axis. The general solution (k) is obtained by the superposition of two such waves of the same shape traveling with the same velocity in two opposite directions. The striking body produces during impact a continuous series of such waves, which travel towards the fixed end and are reflected there. The shape of these consecutive waves can now be established by using the initial conditions and the terminal condition at the end $x = l$.

For the initial moment ($t = 0$) we have, from eq. (k),

$$(u)_{t=0} = f(-x) - f(+x),$$

$$\left(\frac{\partial u}{\partial x}\right)_{t=0} = -f'(-x) - f'(+x),$$

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = af'(-x) - af'(+x).$$

Now by using the initial conditions (f) we obtain,

$$\begin{aligned} -f'(-x) - f'(+x) &= 0 \quad \text{for} \quad 0 < x < l, \\ f'(-x) - f'(+x) &= 0 \quad \text{for} \quad 0 < x < l. \end{aligned} \tag{l}$$

Considering f as a function of an argument z, which can be put equal to $+x$ or $-x$, it can be concluded, from (l), that when $-l < z < l$, $f'(z)$ is equal to zero, since only under this condition both equations (l) can be satisfied simultaneously and hence $f(z)$ is a constant which can be taken equal to zero and we get,

$$f(z) = 0 \quad \text{when} \quad -l < z < l. \tag{m}$$

Now the values of the function $f(z)$ can be determined for the values of z outside the interval $-l < z < l$ by using the end condition (e).

Substituting (k) in eq. (e) we obtain

$$ml\{f''(at - l) - f''(at + l)\} = +f'(at - l) + f'(at + l)$$

or by putting $at + l = z$,

$$f''(z) + \frac{1}{ml}f'(z) = f''(z - 2l) - \frac{1}{ml}f'(z - 2l). \tag{n}$$

By using this equation the function $f(z)$ can be constructed step by step as follows:

From (m) we know that in the interval $l < z < 3l$ the right hand member of equation (n) is zero. By integrating this equation the function $f(z)$ in the interval $l < z < 3l$ will be obtained. The right hand member of equation (n) will then become known for the interval $3l < z < 5l$. Consequently the integration of this equation will give the function $f(z)$ for the interval $3l < z < 5l$. By proceeding in this way the function $f(z)$ can be determined for all values of z greater than $-l$.

Considering eq. (n) as an equation to determine $f'(z)$ the general solution of this linear equation of the first order will be

$$f'(z) = Ce^{-z/ml} + e^{-z/ml}\int e^{z/ml}\left(f''(z - 2l) - \frac{1}{ml}f'(z - 2l)\right)dz, \tag{p}$$

in which C is a constant of integration.

For the interval $l < z < 3l$, the right hand member of eq. (n) vanishes and we obtain

$$f'(z) = Ce^{-z/ml}.$$

Now, by using the condition (g), we have

$$a\{f'(-l+0) - f'(l+0)\} = -v$$

or

$$f'(l+0) = Ce^{-l/m} = \frac{v}{a}; \qquad C = e^{l/m}\frac{v}{a};$$

and we obtain for the interval $l < z < 3l$

$$f'(z) = \frac{v}{a}e^{-(z-l)/ml}. \tag{q}$$

When $3l < z < 5l$, we have, from eq. (q)

$$f'(z-2l) = \frac{v}{a}e^{-(z-3l)/ml}$$

and

$$f''(z-2l) - \frac{1}{ml}f'(z-2l) = -\frac{2}{ml}\frac{v}{a}e^{-(z-3l)/ml}.$$

Now the solution (p) can be represented in the following form,

$$f'(z) = Ce^{-z/ml} - \frac{2}{ml}\frac{v}{a}(z-3l)e^{-(z-3l)/ml}. \tag{r}$$

The constant of integration C will be determined from the condition of continuity of the velocity at the end $x = l$ at the moment $t = (2l/a)$. This condition is

$$\left(\frac{\partial u}{\partial t}\right)_{\substack{t=2l/a-0\\x=l}} = \left(\frac{\partial u}{\partial t}\right)_{\substack{t=2l/a+0\\x=l}}$$

or by using eq. (k)

$$f'(l-0) - f'(3l-0) = f'(l+0) - f'(3l+0).$$

Using now eqs. (m) (q) and (r) we obtain

$$-\frac{v}{a}e^{-2/m} = \frac{v}{a} - Ce^{-3/m},$$

from which

$$C = \frac{v}{a}(e^{l/m} + e^{3/m})$$

and we have for the interval $3l < z < 5l$

$$f'(z) = \frac{v}{a}e^{-(z-l)/ml} + \frac{v}{a}\left(1 - \frac{2}{ml}(z-3l)\right)e^{-(z-3l)/ml}. \tag{s}$$

Knowing $f'(z)$ when $3l < z < 5l$ and using eq. (n), the expression for $f'(z)$ when $5l < z < 7l$ can be obtained and so on.

The function $f(z)$ can be determined by integration if the function $f'(z)$ be known, the constant of integration being determined from the condition that there is no abrupt change in the displacement u at $x = l$. In this manner the following results are obtained when $l < z < 3l$

$$f(z) = mlv/a\{1 - e^{-(z-l)/ml}\} \tag{t}$$

when $3l < z < 5l$

$$f(z) = -\frac{mlv}{a}e^{-(z-l)/ml} + \frac{mlv}{a}\left(1 + \frac{2}{ml}(z - 3l)\right)e^{-(z-3l)/ml}. \tag{v}$$

Knowing $f(z)$ the displacements and the stresses at any cross section of the bar can be calculated by substituting in eq. (k) the corresponding values for the functions $f(at - x)$ and $f(at + x)$. When $0 < t < (l/a)$ the term $f(at - x)$ in eq. (k) is equal to zero, by virtue of (m) and hence we have only the wave $f(at + x)$ advancing in the negative direction of the x axis. The shape of this wave will be obtained from (t) by substituting $at + x$ for z. At $t = (l/a)$ this wave will be reflected from the fixed end and in the interval $(l/a) < t < (2l/a)$ we will have two waves, the wave $f(at - x)$ traveling in the positive direction along the x axis and the wave $f(at + x)$ traveling in the negative direction. Both waves can be obtained from (t) by substituting, for z, the arguments $(at - x)$ and $(at + x)$, respectively. Continuing in this way the complete picture of the phenomenon of longitudinal impact can be secured.

The above solution represents the actual conditions only as long as there exists a positive pressure between the striking body and the bar, i.e., as long as the unit elongation

$$\left(\frac{\partial u}{\partial x}\right)_{x=l} = -f'(at - l) - f'(at + l) \tag{w}$$

remains negative. When $0 < at < 2l$, the right hand member of the eq. (w) is represented by the function (q) with the negative sign and remains negative. When $2l < at < 4l$ the right side of the eq. (w) becomes

$$-\frac{v}{a}e^{-at/ml}\left\{1 + 2e^{2/m}\left(1 - \frac{at - 2l}{ml}\right)\right\}.$$

This vanishes when

$$1 + 2e^{2/m}\left(1 - \frac{at - 2l}{ml}\right) = 0$$

or

$$2at/ml = 4/m + 2 + e^{-2/m}. \tag{x}$$

This equation can have a root in the interval, $2l < at < 4l$ only if,

$$2 + e^{-2/m} < 4/m,$$

which happens for $m = 1.73$.

Hence, if the ratio of the weight of the striking body to the weight of the bar is less than 1.73 the impact ceases at an instant in the interval $2l < at < 4l$ and this instant can be calculated from the eq. (x). For larger values of the ratio m, an investigation of whether or not the impact ceases at some instant in the interval $4l < at < 6l$ should be made, and so on.

The maximum compressive stresses during impact occur at the fixed end and for large values of m ($m > 24$) can be calculated with sufficient accuracy from the following approximate formula:

$$p_{\max} = E\frac{v}{a}(\sqrt{m} + 1). \tag{183}$$

For comparison it is interesting to note that by using the approximate method of the previous paragraph and neglecting δ_{st} in comparison with h in eq. (d) (see p. 280) we arrive at the equation

$$p_{\max} = E\frac{v}{a}\sqrt{m}. \tag{184}$$

When $5 < m < 24$ the equation

$$p_{\max} = E\frac{v}{a}(\sqrt{m} + 1.1) \tag{185}$$

should be used instead of eq. (183). When $m < 5$, S. Venant derived the following approximate formula,

$$p_{\max} = 2E\frac{v}{a}(1 + e^{-2/m}). \tag{186}$$

By using the above method the case of a rod free at one end and struck longitudinally at the other and the case of longitudinal impact of two prismatical bars can be considered.* It should be noted that the investigation of the longitudinal impact given above is based on the assumption that the surfaces of contact between the striking body and the bar are two ideal smooth parallel planes. In actual conditions, there will always be some surface irregularities and a certain interval of time is required to flatten down the high spots. If this interval is of the same

* See A. E. H. Love, loc. cit., p. 435.

order as the time taken for a sound wave to pass along the bar, a satisfactory agreement between the theory and experiment cannot be expected.* Much better results will be obtained if the arrangement is such that the time l/a is comparatively long. For example, by replacing the solid bar by a helical spring C. Ramsauer obtained † a very good agreement between theory and experiment. For this reason we may also expect satisfactory results in applying the theory to the investigation of the propagation of impact waves in long uniformly loaded railway trains. Such a problem may be of practical importance in studying the forces acting in couplings between cars.

Another method of obtaining better agreement between theory and experiment is to make the contact conditions more definite. By taking, for instance, a bar with a rounded end and combining the Herz theory for the local deformation at the point of contact with S. Venant's theory of the waves traveling along the bar, J. E. Sears ‡ secured a very good coincidence between theoretical and experimental results.

54. Vibration of a Circular Ring.—The problem of the vibration of a circular ring is encountered in the investigation of the frequencies of vibration of various kinds of circular frames for rotating electrical machinery as is necessary in a study of the causes of noise produced by such machinery. In the following, several simple problems on the vibration of a circular ring of constant cross section are considered, under the assumptions that the cross sectional dimensions of the ring are small in comparison with the radius of its center line and that one of the principal axes of the cross section is situated in the plane of the ring.

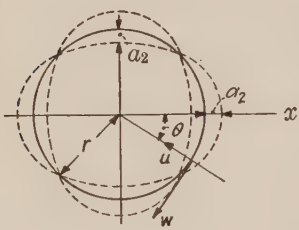


Fig. 134.

Pure Radial Vibration.—In this case the center line of the ring forms a circle of periodically varying radius and all the cross sections move radially without rotation.

Let r = radius of the center line of the ring,
u = the radial displacement, the same for all cross sections.
A = the cross sectional area of the ring.

* Such experiments with solid steel bars were made by W. Voigt, Wied. Ann., Vol. 19, p. 43 (1883).

† Ann. d. Phys., Vol. 30 (1909).

‡ Trans. Cambridge Phil. Soc., Vol. 21 (1908), p. 49. Further experiments are described by J. E. P. Wagstaff, London, Royal Soc. Proc. (ser. A), Vol. 105, p. 544 (1924).

The unit elongation of the ring in the circumferential direction is then $-u/r$. The potential energy of deformation, consisting in this case of the energy of simple tension will be given by the equation:

$$V = \frac{AEu^2}{2r^2} 2\pi r, \tag{a}$$

while the kinetic energy of vibration will be

$$T = \frac{A\gamma}{2g} \dot{u}^2 2\pi r. \tag{b}$$

From (a) and (b) we obtain

$$\ddot{u} + \frac{Eg}{\gamma} \frac{1}{r^2} u = 0,$$

from which

$$u = C_1 \cos pt + C_2 \sin pt,$$

where

$$p = \sqrt{\frac{Eg}{\gamma r^2}}.$$

The frequency of pure radial vibration is therefore *

$$f = \frac{p}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{Eg}{\gamma r^2}}. \tag{187}$$

A circular ring possesses also modes of vibration analogous to the longitudinal vibrations of prismatical bars. If i denotes the number of wave lengths to the circumference, the frequencies of the higher modes of extensional vibration of the ring will be determined from the equation,†

$$f_i = \frac{1}{2\pi} \sqrt{\frac{Eg}{\gamma r^2}} \sqrt{1 + i^2}. \tag{188}$$

Torsional Vibration.— Consideration will now be given to the simplest mode of torsional vibration, i.e., that in which the center line of the ring remains undeformed and all the cross sections of the ring rotate during vibration through the same angle (Fig. 135). Due to this rotation a point M, distant y from the middle plane of the ring,

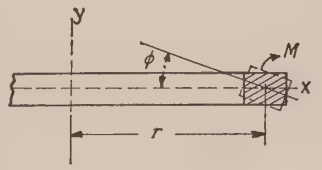


FIG. 135.

* If there is any additional load, which can be considered as uniformly distributed along the center line of the ring, it is only necessary in the above calculation (eq. b) to replace $A\gamma$ by $A\gamma + q$, where q denotes the additional weight per unit length of the center line of the ring.

† See A. E. H. Love, loc. cit., p. 454.

will have a radial displacement equal to $y\varphi$ and the corresponding circumferential elongation can be taken approximately equal to $y\varphi/r$. The potential energy of deformation of the ring can now be calculated as follows:

$$V = 2\pi r \int_A \frac{E}{2}\left(\frac{y\varphi}{r}\right)^2 dA = \frac{\pi E I_x \varphi^2}{r}, \tag{c}$$

where I_x = moment of inertia of the cross section about the x axis.

The kinetic energy of vibration will be

$$T = 2\pi r \cdot \frac{I_p \gamma}{2g}\dot{\varphi}^2, \tag{d}$$

where I_p is the polar moment of inertia of the cross section.

From (c) and (d) we obtain

$$\ddot{\varphi} + \frac{Eg}{\gamma r^2}\frac{I_x}{I_p}\varphi = 0,$$

from which

$$\varphi = C_1 \cos pt + C_2 \sin pt,$$

where

$$p = \sqrt{\frac{Eg}{\gamma r^2}\frac{I_x}{I_p}}.$$

The frequency of torsional vibration will then be given by

$$f = \frac{1}{2\pi}\sqrt{\frac{Eg}{\gamma r^2}\frac{I_x}{I_p}}. \tag{189}$$

Comparing this result with formula (187) it can be concluded that the frequencies of the torsional and pure radial vibrations are in the ratio $\sqrt{I_x/I_p}$. The frequencies of the higher modes of torsional vibration are given,* in the case of a circular cross section of the ring, by the equation,

$$f_i = \frac{1}{2\pi}\sqrt{\frac{Eg}{2\gamma r^2}}\sqrt{1 + i^2}. \tag{190}$$

Remembering that

$$\sqrt{\frac{Eg}{\gamma r^2}} = \frac{a}{r},$$

where a is the velocity of propagation of sound along the bar, it can be concluded that the extensional and torsional vibrations considered above have usually high frequencies. Much lower frequencies will be obtained if flexural vibrations of the ring are considered.

* See A. E. H. Love, loc. cit., p. 453.

Flexural Vibrations of a Circular Ring.—Flexural vibrations of a circular ring fall into two classes, i.e., flexural vibrations in the plane of the ring and flexural vibrations involving both displacements at right angles to the plane of the ring and twist.* In considering the flexural vibrations in the plane of the ring (Fig. 134) let

θ = the angle determining the position of a point on the center line.
u = radial displacement, positive in the direction towards the center.
w = tangential displacement, positive in the direction of the increase in the angle θ.
I = moment of inertia of the cross section with respect to a principal axis at right angles to the plane of the ring.

The unit elongation of the center line at any point, due to the displacements u and w is,

$$e = -\frac{u}{r} + \frac{\partial w}{r\partial\theta} \tag{e}$$

and the change in curvature can be represented by the equation †

$$\frac{1}{r + \Delta r} - \frac{1}{r} = \frac{\partial^2 u}{r^2 \partial\theta^2} + \frac{u}{r^2}. \tag{f}$$

In the most general case of flexural vibration the radial displacement u can be represented in the form of a trigonometrical series ‡

$$u = a_1 \cos\theta + a_2 \cos 2\theta + \cdots + b_1 \sin\theta + b_2 \sin 2\theta + \cdots \tag{h}$$

in which the coefficients $a_1, a_2, \cdots b_1, b_2, \cdots$, varying with the time, represent the generalized coordinates.

Considering flexural vibrations without extension, we have, from (e),

$$u = \frac{\partial w}{\partial\theta}, \tag{g}$$

from which,§

$$w = a_1 \sin\theta + \tfrac{1}{2}a_2 \sin 2\theta + \cdots - b_1 \cos\theta - \tfrac{1}{2}b_2 \cos 2\theta - \cdots. \tag{k}$$

The bending moment at any cross section of the ring will be

$$M = \frac{EI}{r^2}\left(\frac{\partial^2 u}{\partial\theta^2} + u\right),$$

* A. E. H. Love, loc. cit., p. 451.

† This equation was established by J. Boussinesq: Comptes Rendus. Vol. 97 p. 843 (1883).

‡ The constant term of the series, corresponding to pure radial vibration, is omitted.

§ The constant of integration representing a rotation of the ring in its plane as a rigid body, is omitted in the expression (k).

and hence we obtain for the potential energy of bending

$$V = \frac{EI}{2r^4}\int_0^{2\pi}\left(\frac{\partial^2 u}{\partial\theta^2} + u\right)^2 r d\theta,$$

or, by substituting the series (h) for u and by using the formulae,

$$\int_0^{2\pi}\cos m\theta \cos n\theta d\theta = 0, \qquad \int_0^{2\pi}\sin m\theta \sin n\theta d\theta = 0, \qquad \text{when} \qquad m \neq n,$$

$$\int_0^{2\pi}\cos m\theta \sin m\theta d\theta = 0, \qquad \int_0^{2\pi}\cos^2 m\theta d\theta = \int_0^{2\pi}\sin^2 m\theta d\theta = \pi,$$

we get

$$V = \frac{EI\pi}{2r^3}\sum_{i=1}^{i=\infty}(1 - i^2)^2(a_i^2 + b_i^2). \tag{l}$$

The kinetic energy of the vibrating ring is

$$T = \frac{A\gamma}{2g}\int_0^{2\pi}(\dot{u}^2 + \dot{w}^2) r d\theta.$$

By substituting (h) and (k) for u and w, this becomes

$$T = \frac{\pi r A\gamma}{2g}\sum_{i=1}^{i=\infty}\left(1 + \frac{1}{i^2}\right)(\dot{a}_i^2 + \dot{b}_i^2). \tag{m}$$

It is seen that the expressions (l) and (m) contain only the squares of the generalized coordinates and of the corresponding velocities; hence these coordinates are the *principal or normal coordinates* and the corresponding vibrations are the principal modes of flexural vibration of the ring. The differential equation for any mode of vibration, from (l) and (m), will be

$$\frac{\pi r A\gamma}{g}\left(1 + \frac{1}{i^2}\right)\ddot{a}_i + \frac{EI\pi}{r^3}(1 - i^2)^2 a_i = 0$$

or

$$\ddot{a}_i + \frac{Eg}{\gamma}\frac{I}{Ar^4}\frac{i^2(1 - i^2)^2}{1 + i^2}a_i = 0.$$

Hence the frequency of any mode of vibration is determined by the equation:

$$f_i = \frac{1}{2\pi}\sqrt{\frac{Eg}{\gamma}\frac{I}{Ar^4}\frac{i^2(1 - i^2)^2}{1 + i^2}}. \tag{191}$$

When $i = 1$, we obtain $f_1 = 0$. In this case $u = a_1 \cos\theta$; $w = a_1 \sin\theta$

and the ring moves as a rigid body, a_1 being the displacement in the negative direction of the x axis. When $i = 2$ the ring performs the fundamental mode of flexural vibration. The extreme positions of the ring during this vibration are shown in Fig. 134 by dotted lines.

In the case of flexural vibrations of a ring of circular cross section involving both displacements at right angles to the plane of the ring and twist the frequencies of the principal modes of vibration can be calculated from the equation*

$$f_i = \frac{1}{2\pi}\sqrt{\frac{Eg}{\gamma}\,\frac{I}{Ar^4}\,\frac{i^2(i^2-1)^2}{i^2+1+\sigma}}, \tag{192}$$

in which σ denotes Poisson's ratio.

Comparing (192) and (191) it can be concluded that even in the lowest mode ($i = 2$) the frequencies of the two classes of flexural vibrations differ but very slightly.

Incomplete Ring.—When the ring has the form of an incomplete circular arc, the problem of the calculation of the natural frequencies of vibration becomes very complicated.* The results so far obtained can be interpreted only for the case where the length of the arc is small in comparison to the radius of curvature. In such cases, these results show that natural frequencies are slightly lower than those of a straight bar of the same material, length, and cross section. Since, in the general case, the exact solution of the problem is extremely complicated, at this date only some approximate values for the lowest natural frequency are available, the Rayleigh-Ritz method † being used in their calculation.

55. Vibration of Membranes.—*General.*—In the following discussion it is assumed that the membrane is a perfectly flexible and infinitely thin lamina of uniform material and thickness. It is further assumed that it is uniformly stretched in all directions by a tension so large that the fluctuation in this tension due to the small deflections during vibration can be neglected. Taking the plane of the membrane coinciding with the xy plane, let

w = the displacement of any point of the membrane at right angles to the xy plane during vibration.

s = uniform tension per unit length of the boundary.

q = weight of the membrane per unit area.

*A. E. H. Love, Mathematical Theory of Elasticity, 4th Ed., Cambridge, 1927, p. 453.

† This problem has been discussed by H. Lamb, London Math. Soc. Proc., Vol. 19, p. 365 (1888).

‡ See J. P. DenHartog, The Lowest Natural Frequency of Circular Arcs, Phil. Mag., Vol. 5 (1928), p. 400; also: Vibration of Frames of Electrical Machines, Trans. A. S. M. E. Applied Mech. Div. 1928.

The increase in the potential energy of the membrane during deflection will be found in the usual way by multiplying the uniform tension s by the increase in surface area of the membrane. The area of the surface of the membrane in a deflected position will be

$$A = \int\int \sqrt{1 + \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2}\, dxdy$$

or, noting that the deflections during vibration are very small,

$$A = \int\int \left\{1 + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right\} dxdy.$$

Then the increase in potential energy will be

$$V = \frac{s}{2}\int\int \left\{\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right\} dxdy. \qquad (a)$$

The kinetic energy of the membrane during vibration is

$$T = \frac{q}{2g}\int\int \dot{w}^2 dxdy. \qquad (b)$$

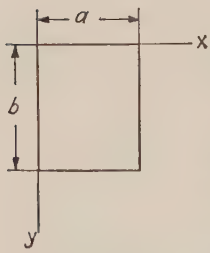


Fig. 136.

By using (a) and (b) the frequencies of the normal modes of vibration can be calculated as will now be shown for some particular cases.

Vibration of a Rectangular Membrane.—Let a and b denote the lengths of the sides of the membrane and let the axes be taken as shown in Fig. 136. Whatever function of the coordinates w may be, it always can be represented within the limits of the rectangle by the double series

$$w = \sum_{m=1}^{m=\infty} \sum_{n=1}^{n=\infty} q_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}, \qquad (c)$$

the coefficients q_{mn} of which are taken as the generalized coordinates for this case. It is easy to see that each term of the series (c) satisfies the boundary conditions, namely, $w = 0$, for $x = 0$; $x = a$ and $w = 0$ for $y = 0$; $y = b$.

Substituting (c) in the expression (a) for the potential energy we obtain

$$V = \frac{s\pi^2}{2}\int_0^a\int_0^b \left\{\left(\sum\sum q_{mn}\frac{m}{a}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\right)^2 + \right.$$
$$\left. + \left(\sum\sum q_{mn}\frac{n}{b}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\right)^2\right\} dxdy.$$

Integrating this expression over the area of the membrane using the formulae of paragraph (12) (see p. 47) we find,

$$V = \frac{s}{2}\frac{ab\pi^2}{4}\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)q_{mn}^2. \tag{d}$$

In the same way by substituting (*c*) in eq. (*b*) the following expression for the kinetic energy will be obtained:

$$T = \frac{q}{2g}\frac{ab}{4}\sum\sum \dot{q}_{mn}^2. \tag{e}$$

The expressions (*d*) and (*e*) do not contain the products of the coordinates and of the corresponding velocities, hence the coordinates chosen are principal coordinates and the corresponding vibrations are normal modes of vibration of the membrane.

The differential equation of a normal vibration, from (*d*) and (*e*), will be

$$\frac{q}{g}\frac{ab}{4}\ddot{q}_{mn} + s\frac{ab\pi^2}{4}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)q_{mn} = 0, \tag{f}$$

from which,

$$f_{mn} = \frac{1}{2}\sqrt{\frac{gs}{q}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)}. \tag{193}$$

The lowest mode of vibration will be obtained by putting $m = n = 1$. Then

$$f_{11} = \frac{1}{2}\sqrt{\frac{gs}{q}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}.$$

The deflection surface of the membrane in this case is

$$w = C\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}. \tag{g}$$

In the same manner the higher modes of vibration can be obtained. Take, for instance, the case of a square membrane, when $a = b$. The frequency of the lowest tone is

$$f_{11} = \frac{1}{a\sqrt{2}}\sqrt{\frac{gs}{q}}. \tag{194}$$

The frequency is directly proportional to the square root of the tension s and inversely proportional to the length of sides of the membrane and to the square root of the load per unit area.

The next two higher modes of vibration will be obtained by taking one of the numbers m, n equal to 2 and the other to 1. These two modes have the same frequency, but show different shapes of deflection surface. In Fig. 137, a and b the node lines of these two modes of vibration are shown. Because of the fact that the frequencies are the same it is possible to superimpose these two surfaces on each other in any ratio of their maximum deflections. Such a combination is expressed by

$$w = \left(C \sin \frac{2\pi x}{a} \sin \frac{\pi y}{a} + D \sin \frac{\pi x}{a} \sin \frac{2\pi y}{a} \right),$$

where C and D are arbitrary quantities. Four particular cases of such a combined vibration are shown in Fig. 137. Taking $D = 0$ we obtain

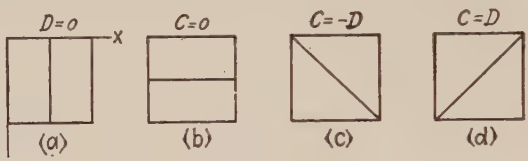


FIG. 137.

the vibration mentioned above and shown in Fig. 137, a. The membrane, while vibrating, is sub-divided into two equal parts by a vertical nodal line. When $C = 0$, the membrane is sub-divided by a horizontal nodal line as in Fig. 137, b. When $C = D$, we obtain

$$w = C\left(\sin \frac{2\pi x}{a} \sin \frac{\pi y}{a} + \sin \frac{\pi x}{a} \sin \frac{2\pi y}{a} \right)$$
$$= 2C \sin \frac{\pi x}{a} \sin \frac{\pi y}{a} \left(\cos \frac{\pi x}{a} + \cos \frac{\pi y}{a} \right).$$

This expression vanishes when

$$\sin \frac{\pi x}{a} = 0, \qquad \text{or} \qquad \sin \frac{\pi y}{a} = 0$$

or again when

$$\cos \frac{\pi x}{a} + \cos \frac{\pi y}{a} = 0.$$

The first two equations give us the sides of the boundary; from the third equation we obtain

$$\frac{\pi x}{a} = \pi - \frac{\pi y}{a}$$

or

$$x + y = a.$$

This represents one diagonal of the square shown in Fig. 137, *d*. Fig. 137, *c* represents the case when $C = -D$. Each half of the membrane in the last two cases can be considered as an isosceles right-angled triangular membrane. The fundamental frequency of this membrane, from eq. (193), will be

$$f = \frac{1}{2}\sqrt{\frac{gs}{q}\left(\frac{4}{a^2}+\frac{1}{a^2}\right)} = \frac{\sqrt{5}}{2a}\sqrt{\frac{gs}{q}}. \tag{195}$$

In this manner also higher modes of vibration of a square or rectangular membrane can be considered.*

In the case of forced vibration of the membrane the differential equation of motion (f) becomes

$$\frac{q}{g}\frac{ab}{4}\ddot{q}_{mn} + s\frac{ab\pi^2}{4}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)q_{mn} = Q_{mn}, \tag{h}$$

in which Q_{mn} is the *generalized disturbing force* corresponding to the coordinate q_{mn}.

Let us consider, as an example, the case of a harmonic force $P = P_0 \cos \omega t$, acting at the center of the membrane. By giving an increase δq_{mn} to a coordinate q_{mn}, in the expression (c), we find for the work done by the force P:

$$P_0 \cos \omega t \delta q_{mn} \sin\frac{m\pi}{2}\sin\frac{n\pi}{2},$$

from which we see that when m and n are both odd, $Q_{mn} = \pm P_0 \cos \omega t$, otherwise $Q_{mn} = 0$. Substituting in eq. (h), and using eq. 24, (p. 20), we obtain

$$q_{mn} = \pm\frac{4g}{abq}\frac{P_0}{p_{mn}}\int_0^t \sin p_{mn}(t-t_1)\cos\omega t_1 dt_1$$

$$= \pm\frac{4g}{abq}\frac{P_0}{p_{mn}^2-\omega^2}(\cos\omega t - \cos p_{mn}t), \tag{k}$$

where,

$$p_{mn}^2 = \frac{gs\pi^2}{q}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right).$$

By substituting (k) in the expression (c) the vibrations produced by the disturbing force $P_0 \cos \omega t$ will be obtained.

When a distributed disturbing force Z is acting on the membrane, the

* A more detailed discussion of this problem can be found in Rayleigh's book, loc. cit., p. 306. See also Lamé's, Leçons sur l'élasticité. Paris, 1852.

generalized force in eq. (h) becomes

$$Q_{mn} = \int_0^b \int_0^a Z \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dxdy. \tag{l}$$

Assume, for instance, that a uniformly distributed pressure Z is suddenly applied to the membrane at the initial moment ($t = 0$), then from (l),

$$Q_{mn} = Z \frac{ab}{mn\pi^2} (1 - \cos m\pi)(1 - \cos n\pi).$$

When m and n both are odd, we have

$$Q_{mn} = \frac{4ab}{mn\pi^2} Z; \tag{m}$$

otherwise Q_{mn} vanishes.

Substituting (m) in eq. (h) and using the initial condition that $q_{mn} = 0$ at $t = 0$, we obtain

$$q_{mn} = \frac{16g}{qmn\pi^2} \frac{Z(1 - \cos p_{mn}t)}{p_{mn}^2}. \tag{n}$$

Hence the vibrations produced by the suddenly applied pressure Z are

$$w = \frac{16gZ}{\pi^2 q} \sum \sum \frac{1 - \cos p_{mn}t}{mnp_{mn}^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \tag{o}$$

where m and n are both odd.

Rayleigh-Ritz Method.—In calculating the frequencies of the natural modes of vibration of a membrane the Rayleigh-Ritz method is very useful. In applying this method we assume that the deflections of the membrane, while vibrating, are given by

$$w = w_0 \cos pt, \tag{p}$$

where w_0 is a suitable function of the coordinates x and y which determines the shape of the deflected membrane, i.e., the mode of vibration. Substituting (p) in the expression (a) for the potential energy, we find

$$V_{\max} = \frac{s}{2} \int\int \left\{ \left(\frac{\partial w_0}{\partial x} \right)^2 + \left(\frac{\partial w_0}{\partial y} \right)^2 \right\} dxdy. \tag{q}$$

For the maximum kinetic energy we obtain from (b)

$$T_{\max} = \frac{q}{2g} p^2 \int\int w_0^2 dxdy. \tag{r}$$

Putting (q) equal to (r) we get

$$p^2 = \frac{sg}{q} \frac{\iint \left\{ \left(\frac{\partial w_0}{\partial x} \right)^2 + \left(\frac{\partial w_0}{\partial y} \right)^2 \right\} dxdy}{\iint w_0^2 dxdy}. \tag{s}$$

In applying the Rayleigh-Ritz method we take the expression w_0 for the deflection surface of the membrane in the form of a series:

$$w_0 = a_1\varphi_1(x, y) + a_2\varphi_2(x, y) + a_3\varphi_3(x, y) + \cdots, \tag{t}$$

each term of which satisfies the conditions at the boundary. (The deflections at the boundary of the membrane must be equal to zero.) The coefficients a_1, a_2, $\cdots$ in this series should be chosen in such a manner as to make (s) a minimum, i.e., so as to satisfy all equations of the following form

$$\frac{\partial}{\partial a_n} \frac{\iint \left\{ \left(\frac{\partial w_0}{\partial x} \right)^2 + \left(\frac{\partial w_0}{\partial y} \right)^2 \right\} dxdy}{\iint w_0^2 dxdy} = 0,$$

or

$$\iint w_0^2 dxdy \frac{\partial}{\partial a_n} \iint \left\{ \left(\frac{\partial w_0}{\partial x} \right)^2 + \left(\frac{\partial w_0}{\partial y} \right)^2 \right\} dxdy -$$
$$- \iint \left\{ \left(\frac{\partial w_0}{\partial x} \right)^2 + \left(\frac{\partial w_0}{\partial y} \right)^2 \right\} dxdy \frac{\partial}{\partial a_n} \iint w_0^2 dxdy = 0.$$

By using (s) this latter equation becomes,

$$\frac{\partial}{\partial a_n} \iint \left\{ \left(\frac{\partial w_0}{\partial x} \right)^2 + \left(\frac{\partial w_0}{\partial y} \right)^2 - \frac{p^2 q}{gs} w_0^2 \right\} dxdy = 0. \tag{u}$$

In this manner we obtain as many equations of the type (u) as there are coefficients in the series (t). All these equations will be linear in a_1, a_2, a_3, $\cdots$, and by equating the determinant of these equations to zero the frequency equation for the membrane will be obtained.

Considering, for instance, the modes of vibration of a square membrane symmetrical with respect to the x and y axes, Fig. 138, the series (t) can be taken in the following form,

$$w_0 = (a^2 - x^2)(a^2 - y^2)(a_1 + a_2x^2 + a_3y^2 + a_4x^2y^2 + \cdots).$$

It is easy to see that each term of this series becomes equal to zero,

when $x = y = \pm a$. Hence the conditions at the boundary are satisfied.

In the case of a convex polygon the boundary conditions will be satisfied by taking

$$w_0 = (a_1x + b_1y + c_1)(a_2x + b_2y + c_2)\cdots(a_nx + b_ny + c_n) \sum\sum a_{mn}x^my^n,$$

where $a_1x + b_1y + c_1 = 0, \cdots$ are the equations of the sides of the polygon. By taking only the first term ($m = 0$, $n = 0$) of this series a satisfactory approximation for the fundamental type of vibration usually will be obtained. It is necessary to take more terms if the frequencies of higher modes of vibration are required.

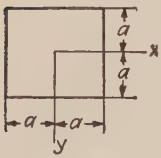


FIG. 138.

Circular Membrane.—We will consider the simplest case of vibration, where the deflected surface of the membrane is symmetrical with respect to the center of the circle. In this case the deflections depend only on the radial distance r and the boundary condition will be satisfied by taking

$$w_0 = a_1 \cos\frac{\pi r}{2a} + a_2 \cos\frac{3\pi r}{2a} + \cdots, \qquad (v)$$

where a denotes the radius of the boundary.

Because we are using polar coordinates, equation (q) has to be replaced in this case by the following equation:

$$V_{\max} = \frac{s}{2}\int_0^a \left(\frac{\partial w_0}{\partial r}\right)^2 2\pi r dr. \qquad (q)^1$$

Instead of (r) we obtain

$$T_{\max} = \frac{q}{2g}p^2 \int_0^a w_0^2 2\pi r dr \qquad (r)^1$$

and eq. (u) assumes the form

$$\frac{\partial}{\partial a_n}\int_0^a \left\{\left(\frac{\partial w_0}{\partial r}\right)^2 - \frac{p^2q}{gs}w_0^2\right\} 2\pi r dr = 0. \qquad (u)^1$$

By taking only the first term in the series (v) and substituting $w_0 = a_1 \cos \pi r/2a$ in eq. $(u)^1$ we obtain

$$\frac{\pi^2}{4a^2}\int_0^a \sin^2\frac{\pi r}{2a} r dr = \frac{p^2q}{gs}\int_0^a \cos^2\frac{\pi r}{2a} r dr,$$

from which

$$\frac{\pi^2}{4a^2}\left(\frac{1}{2} + \frac{2}{\pi^2}\right) = \frac{p^2q}{gs}\left(\frac{1}{2} - \frac{2}{\pi^2}\right)$$

or

$$p = \frac{2.415}{a}\sqrt{\frac{gs}{q}}.$$

The exact solution * gives for this case,

$$p = \frac{2.404}{a}\sqrt{\frac{gs}{q}}. \tag{196}$$

The error of the first approximation is less than ½%.

In order to get a better approximation for the fundamental note and also for the frequencies of the higher modes of vibration, a larger number of terms in the series (v) should be taken. These higher modes of vibration will have one, two, three, $\cdots$ nodal circles at which the displacements w are zero during vibration.

In addition to the modes of vibration symmetrical with respect to the center a circular membrane may have also modes in which one, two, three, $\cdots$ diameters of the circle are *nodal lines*, along which the deflections during vibration are zero. Several modes of vibration of a circular membrane are shown in Fig. 139 where the nodal circles and nodal diameters are indicated by dotted lines.

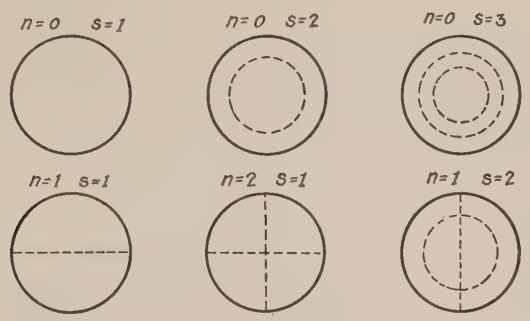


Fig. 139.

In all cases the quantity p, determining the frequencies, can be expressed by the equation,

$$p_{ns} = \frac{\alpha_{ns}}{a}\sqrt{\frac{gs}{q}}. \tag{197}$$

the constants α_{ns} of which are given in the table below.† In this table

* The problem of the vibration of a circular membrane is discussed in detail by Lord Rayleigh, loc. cit., p. 318.

† The table was calculated by Bourget, Ann. de. l'école normale, Vol. 3 (1866).

n denotes the number of nodal diameters and s the number of nodal circles. (The boundary circle is included in this number.)

s	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
1	2.404	3.832	5.135	6.379	7.586	8.780
2	5.520	7.016	8.417	9.760	11.064	12.339
3	8.654	10.173	11.620	13.017	14.373	15.700
4	11.792	13.323	14.796	16.224	17.616	18.982
5	14.931	16.470	17.960	19.410	20.827	22.220
6	18.071	19.616	21.117	22.583	24.018	25.431
7	21.212	22.760	24.270	25.749	27.200	28.628
8	24.353	25.903	27.421	28.909	30.371	31.813

It is assumed in the previous discussion that the membrane has a complete circular area and that it is fixed only on the circular boundary, but it is easy to see that the results obtained include also the solution of other problems such as membranes bounded by two concentric circles and two radii or membranes in the form of a sector. Take, for instance, membrane semi-circular in form. All possible modes of vibration of this membrane will be included in the modes which the circular membrane may perform. It is only necessary to consider one of the nodal diameters of the circular membrane as a fixed boundary. When the boundary of a membrane is approximately circular, the lowest tone of such a membrane is nearly the same as that of circular membrane having the same area and the same value of sg/q. Taking the equation determining the frequency of the fundamental mode of vibration of a membrane in the form,

$$p = \alpha\sqrt{\frac{gs}{qA}}, \tag{198}$$

where A = the area of the membrane, the constant α of this equation will be given by the table on page 307, which shows the effect of a greater or less departure from the circular form.*

In cases where the boundary is different from those discussed above the investigation of the vibrations presents mathematical difficulties and only the case of an elliptical boundary has been completely solved by Mathieu.† A complete discussion of the theory of vibration of membranes from a mathematical point of view is given in a book by Pockels.‡

* The table is taken from Rayleigh's book, loc. cit., p. 345.

† Journal de Math. (Liouville), Vol. 13 (1868).

‡ Pockels: Über die partielle Differentialgleichung, $\Delta u + k^2u = 0$; Leipzig. 1891.

Circle	$\alpha = 2.404\sqrt{\pi}$	$= 4.261$
Square	$\alpha = \sqrt{2}\pi$	$= 4.443$
Quadrant of a circle	$\alpha = \frac{5.135}{2}\sqrt{\pi}$	$= 4.551$
Sector of a circle 60°	$\alpha = 6.379\sqrt{\frac{\pi}{6}}$	$= 4.616$
Rectangle 3×2	$\alpha = \sqrt{\frac{13}{6}\cdot\pi}$	$= 4.624$
Equilateral triangle	$\alpha = 2\pi\sqrt{\tan 30^\circ}$	$= 4.774$
Semi-circle	$\alpha = 3.832\sqrt{\frac{\pi}{2}}$	$= 4.803$
Rectangle 2×1	$\alpha = \pi\sqrt{\frac{5}{2}}$	$= 4.967$
Rectangle 3×1	$\alpha = \pi\sqrt{\frac{10}{3}}$	$= 5.736$

56. Vibration of Plates.—*General.*—In the following discussion it is assumed that the plate consists of a perfectly elastic, homogeneous, isotropic material and that it has a uniform thickness h considered small in comparison with its other dimensions. We take for the xy plane the middle plane of the plate and assume that with small deflections * the lateral sides of an element, cut out from the plate by planes parallel to the zx and zy planes (see Fig. 140) remain plane and rotate so as to be normal to the deflected middle surface of the plate. Then the strain in a thin layer of this element, indicated by the shaded area and distant z from the middle plane can be obtained from a simple geometrical consideration and will be represented by the following equations: †

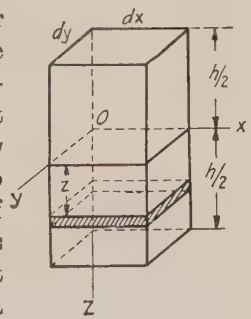


Fig. 140.

$$e_{xx} = \frac{z}{R_1} = -z\frac{\partial^2 w}{\partial x^2},$$

$$e_{yy} = \frac{z}{R_2} = -z\frac{\partial^2 w}{\partial y^2}, \qquad (a)$$

$$e_{xy} = -2z\frac{\partial^2 w}{\partial x \partial y},$$

* The deflections are assumed to be small in comparison with the thickness of the plate.

† It is assumed that there is no stretching of the middle plane.

in which

$$
\begin{aligned}
e_{xx}, e_{yy} &= \text{unit elongations in the } x \text{ and } y \text{ directions,}\\
e_{xy} &= \text{shear deformation in the } xy \text{ plane,}\\
w &= \text{deflection of the plate,}\\
\frac{1}{R_1}, \frac{1}{R_2} &= \text{curvatures in the } xz \text{ and } yz \text{ planes,}\\
h &= \text{thickness of the plate.}
\end{aligned}
$$

The corresponding stresses will then be obtained from the known equations:

$$
\begin{aligned}
p_x &= \frac{E}{1-\sigma^2}(e_{xx} + \sigma e_{yy}) = -\frac{Ez}{1-\sigma^2}\left(\frac{\partial^2 w}{\partial x^2} + \sigma\frac{\partial^2 w}{\partial y^2}\right),\\
p_y &= \frac{E}{1-\sigma^2}(e_{yy} + \sigma e_{xx}) = -\frac{Ez}{1-\sigma^2}\left(\frac{\partial^2 w}{\partial y^2} + \sigma\frac{\partial^2 w}{\partial x^2}\right),\\
p_t &= Ge_{xy} = -\frac{Ez}{(1+\sigma)}\cdot\frac{\partial^2 w}{\partial x\partial y},
\end{aligned}
\tag{b}
$$

in which σ denotes Poisson's ratio.

The potential energy accumulated in the shaded layer of the element during the deformation will be

$$
dV = \left(\frac{e_{xx}p_x}{2} + \frac{e_{yy}p_y}{2} + \frac{e_{xy}p_t}{2}\right) dxdydz
$$

or by using the eqs. (*a*) and (*b*)

$$
\begin{aligned}
dV = \frac{Ez^2}{2(1-\sigma^2)}\Bigg\{&\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2\\
&+ 2\sigma\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2(1-\sigma)\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\Bigg\} dxdydz,
\end{aligned}
\tag{c}
$$

from which, by integration, we obtain the potential energy of bending of the plate

$$
\begin{aligned}
V = \iiint dV = \frac{D}{2}\iint\Bigg\{&\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2\\
&+ 2\sigma\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2(1-\sigma)\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\Bigg\} dxdy,
\end{aligned}
\tag{199}
$$

where $D = \dfrac{Eh^3}{12(1-\sigma^2)}$ is the *flexural rigidity* of the plate.

The kinetic energy of a vibrating plate will be

$$T = \frac{\gamma h}{2g} \int\int \dot{w}^2 dx dy, \tag{200}$$

where $\gamma h/g$ is the mass per unit area of the plate.

From these expressions for V and T, the differential equation of vibration of the plate can be obtained.

Vibration of a Rectangular Plate.—In the case of a rectangular plate (Fig. 136) with simply supported edges we can proceed as in the case of a rectangular membrane and take the deflection of the plate during vibration in the form of a double series

$$w = \sum_{m=1}^{m=\infty} \sum_{n=1}^{n=\infty} q_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}. \tag{d}$$

It is easy to see that each term of this series satisfies the conditions at the edges, which require that w, $\partial^2 w/\partial x^2$ and $\partial^2 w/\partial y^2$ must be equal to zero at the boundary.

Substituting (d) in eq. 199 the following expression for the potential energy will be obtained

$$V = \frac{\pi^4 ab}{8} D \sum_{m=1}^{m=\infty} \sum_{n=1}^{n=\infty} q_{mn}^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2. \tag{201}$$

The kinetic energy will be

$$T = \frac{\gamma h}{2g} \frac{ab}{4} \sum \sum \dot{q}_{mn}^2. \tag{202}$$

It will be noted that the expressions (201) and (202) contain only the squares of the quantities q_{mn} and of the corresponding velocities, from which it can be concluded that these quantities are normal coordinates for the case under consideration. The differential equation of a normal vibration will be

$$\frac{\gamma h}{g} \ddot{q}_{mn} + \pi^4 D q_{mn} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 = 0,$$

from which

$$q_{mn} = C_1 \cos pt + C_2 \sin pt,$$

where

$$p = \pi^2 \sqrt{\frac{gD}{\gamma h}} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right). \tag{203}$$

From this the frequencies of the lowest mode and of the higher modes of

vibration can be easily calculated. Taking, for instance, a square plate we obtain for the lowest mode of vibration

$$f_1 = \frac{p_1}{2\pi} = \frac{\pi}{a^2}\sqrt{\frac{gD}{\gamma h}}. \tag{204}$$

In considering higher modes of vibration and their nodal lines, the discussion previously given for the vibration of a rectangular membrane can be used. Also the case of forced vibrations of a rectangular plate with simply supported edges can be solved without any difficulty. It should be noted that the cases of vibration of a rectangular plate, of which two opposite edges are supported while the other two edges are free or clamped, can also be solved without great mathematical difficulty.*

The problems of the vibration of a rectangular plate, of which all the edges are free or clamped, are, however, much more complicated. For the solution of these problems, Ritz' method has been found to be very useful.† In using this method we assume

$$w = w_0 \cos pt, \tag{e}$$

where w_0 is a function of x and y which determines the mode of vibration. Substituting (*e*) in the equations (199) and (200), the following expressions for the maximum potential and kinetic energy of vibration will be obtained:

$$V_{\max} = \frac{D}{2}\int\int\left\{\left(\frac{\partial^2 w_0}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w_0}{\partial y^2}\right)^2 + 2\sigma\frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w_0}{\partial y^2} + 2(1-\sigma)\left(\frac{\partial^2 w_0}{\partial x \partial y}\right)^2\right\}dxdy$$

$$T_{\max} = \frac{\gamma h}{2g}p^2\int\int w_0^2 dxdy,$$

from which

$$p^2 = \frac{2g}{\gamma h}\frac{V_{\max}}{\int\int w_0^2 dxdy}. \tag{205}$$

Now we take the function w_0 in the form of a series

$$w_0 = a_1\varphi_1(x, y) + a_2\varphi_2(x, y) + \cdots, \tag{f}$$

where φ_1, φ_2, $\cdots$ are suitable functions of x and y, satisfying the con-

* See Voigt, Göttinger Nachrichten, 1893, p. 225.

† See W. Ritz, Annalen der Physik, Vol. 28 (1909), p. 737. See also "Gesammelte Werke" (1911), p. 265.

ditions at the boundary of the plate. It is then only necessary to determine the coefficients $a_1, a_2, \cdots$ in such a manner as to make the right member of (205) a minimum. In this way we arrive at a system of equations of the type:

$$\frac{\partial}{\partial a_n}\iint\left\{\left(\frac{\partial^2 w_0}{\partial x^2}\right)^2+\left(\frac{\partial^2 w_0}{\partial y^2}\right)^2+2\sigma\frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w_0}{\partial y^2}\right.$$
$$\left.+2(1-\sigma)\left(\frac{\partial^2 w_0}{\partial x\partial y}\right)^2-\frac{p^2\gamma h}{gD}w_0^2\right\}dxdy=0. \tag{206}$$

which will be linear with respect to the constants $a_1, a_2, \cdots$ and by equating to zero the determinant of these equations the frequencies of the various modes of vibration can be approximately calculated.

W. Ritz applied this method to the study of the vibration of a square plate with free edges.* The series (f) was taken in this case in the form,

$$w_0=\sum\sum a_{mn}u_m(x)v_n(y), \qquad (f)^1$$

where $u_m(x)$ and $v_n(y)$ are the normal functions of the vibration of a prismatical bar with free ends (see p. 233). The frequencies of the lowest and of the higher modes of vibration will be determined by the equation

$$p=\frac{\alpha}{a^2}\sqrt{\frac{gD}{\gamma h}}. \tag{207}$$

in which α is a constant depending on the mode of vibration. For the three lowest modes the values of this constant are

$$\alpha_1=14.10, \qquad \alpha_2=20.56, \qquad \alpha_3=23.91.$$

The corresponding modes of vibration are represented by their nodal lines in Fig. 141 below

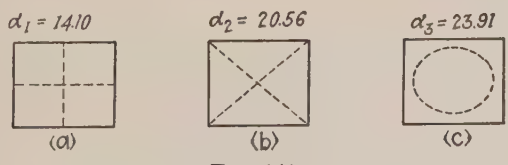


FIG. 141.

An extensive study of the nodal lines for this case and a comparison with experimental data are given in the paper by W. Ritz mentioned above.

* Loc. cit., p. 310.

From eq. (207) some general conclusions can be drawn which hold also in other cases of vibration of plates, namely,

(*a*) The period of the vibration of any natural mode varies with the square of the linear dimensions, provided the thickness remains the same;

(*b*) If all the dimensions of a plate, including the thickness, be increased in the same proportion, the period increases with the linear dimensions;

(*c*) The period varies inversely with the square root of the modulus of elasticity and directly as the square root of the density of material.

Vibration of a Circular Plate.—The problem of the vibration of a circular plate has been solved by G. Kirchhoff * who calculated also the frequencies of several modes of vibration for a plate with free boundary. The exact solution of this problem involves the use of Bessel functions. In the following an approximate solution is developed by means of the Rayleigh-Ritz method, which usually gives for the lowest mode an accuracy sufficient for practical applications. In applying this method it will be useful to transform the expressions (199) and (200) for the potential and kinetic energy to polar coordinates. By taking the coordinates as shown in Fig. 142, we see from the elemental triangle *mns* that by giving to the coordinate x a small increase dx we obtain

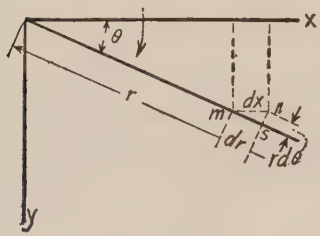


Fig. 142.

$$dr = dx \cos \theta; \qquad d\theta = -\frac{dx \sin \theta}{r}.$$

Then, considering the deflection w as a function of r and θ we obtain,

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial w}{\partial r}\cos \theta - \frac{\partial w}{\partial \theta}\frac{\sin \theta}{r}.$$

In the same manner we will find

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\sin \theta + \frac{\partial w}{\partial \theta}\frac{\cos \theta}{r}.$$

* See Journal f. Math. (Crelle), Vol. 40 (1850), or Gesammelte Abhandlungen von G. Kirchhoff, Leipzig 1882, p. 237, or Vorlesungen über math. Physik, Mechanik, Vorlesung 30.

Repeating the differentiation we obtain

$$\frac{\partial^2 w}{\partial x^2} = \left(\frac{\partial}{\partial r}\cos\theta - \frac{\partial}{\partial\theta}\frac{\sin\theta}{r}\right)\left(\frac{\partial w}{\partial r}\cos\theta - \frac{\partial w}{\partial\theta}\frac{\sin\theta}{r}\right).$$

$$= \frac{\partial^2 w}{\partial r^2}\cos^2\theta - 2\frac{\partial^2 w}{\partial\theta\partial r}\frac{\sin\theta\cos\theta}{r} + \frac{\partial w}{\partial r}\frac{\sin^2\theta}{r} + 2\frac{\partial w}{\partial\theta}\frac{\sin\theta\cos\theta}{r^2} + \frac{\partial^2 w}{\partial\theta^2}\frac{\sin^2\theta}{r^2}, \tag{g}$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial r^2}\sin^2\theta + 2\frac{\partial^2 w}{\partial\theta\partial r}\frac{\sin\theta\cos\theta}{r} + \frac{\partial w}{\partial r}\frac{\cos^2\theta}{r} - 2\frac{\partial w}{\partial\theta}\frac{\sin\theta\cos\theta}{r^2} + \frac{\partial^2 w}{\partial\theta^2}\frac{\cos^2\theta}{r^2}, \tag{h}$$

$$\frac{\partial^2 w}{\partial x\partial y} = \frac{\partial^2 w}{\partial r^2}\sin\theta\cos\theta + \frac{\partial^2 w}{\partial r\partial\theta}\frac{\cos 2\theta}{r} - \frac{\partial w}{\partial\theta}\frac{\cos 2\theta}{r^2} - \frac{\partial w}{\partial r}\frac{\sin\theta\cos\theta}{r} - \frac{\partial^2 w}{\partial\theta^2}\frac{\sin\theta\cos\theta}{r^2}, \tag{k}$$

from which we find:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2},$$

$$\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 = \frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right) - \left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial\theta}\right)\right\}^2.$$

Substituting in eq. (199) and taking the origin at the center of the plate, we obtain

$$V = \frac{D}{2}\int_0^{2\pi}\int_0^a \left\{\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)^2 - 2(1-\sigma)\frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right) + 2(1-\sigma)\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial\theta}\right)\right\}^2\right\} r d\theta dr, \tag{208}$$

where a denotes the radius of the plate.

When the deflection of the plate is symmetrical about the center, u will be a function of r only and eq. (208) becomes

$$V = \pi D\int_0^a \left\{\left(\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right)^2 - 2(1-\sigma)\frac{d^2 w}{dr^2}\cdot\frac{1}{r}\frac{dw}{dr}\right\} r dr. \tag{209}$$

In the case of a plate clamped at the edge, the integral

$$\int\int\left[\frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right) - \left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial\theta}\right)\right\}^2\right] r dr d\theta$$

vanishes and we obtain from (208)

$$V = \frac{D}{2}\int_0^{2\pi}\int_0^a \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)^2 r d\theta dr. \tag{210}$$

If the deflection of such a plate is symmetrical about the center, we have

$$V = \pi D \int_0^a \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)^2 r dr. \tag{211}$$

The expression for the kinetic energy in polar coordinates will be

$$T = \frac{\gamma h}{2g}\int_0^{2\pi}\int_0^a \dot{w}^2 r d\theta dr \tag{212}$$

and in symmetrical cases,

$$T = \frac{\pi\gamma h}{g}\int_0^a \dot{w}^2 r dr. \tag{213}$$

By using these expressions for the potential and kinetic energy the frequencies of the natural modes of vibration of a circular plate for various particular cases can be calculated.

Circular Plate Clamped at the Boundary.—The problem of the circular plate clamped at the edges is of practical interest in connection with the application in telephone receivers and other devices. In using the Rayleigh-Ritz method we assume

$$w = w_0 \cos pt, \tag{l}$$

where w_0 is a function of r and θ.

In the case of the lowest mode of vibration the shape of the vibrating plate is symmetrical about the center of the plate and w_0 will be a function of r only. By taking w_0 in the form of a series like

$$w_0 = a_1\left(1 - \frac{r^2}{a^2}\right)^2 + a_2\left(1 - \frac{r^2}{a^2}\right)^3 + \cdots, \tag{m}$$

the condition of symmetry will be satisfied. The conditions at the boundary also will be satisfied because each term of the series (m) together with its first derivative vanishes when $r = a$.

The differential equation (206) in the case under consideration becomes

$$\frac{\partial}{\partial a_n}\int_0^a \left\{\left(\frac{d^2 w_0}{dr^2} + \frac{1}{r}\frac{dw_0}{dr}\right)^2 - \frac{p^2\gamma h}{gD} w_0^2\right\} r dr = 0. \tag{214}$$

By taking only one term of the series (m) and substituting it in (214) we obtain,

$$\frac{96}{9a^2} - \frac{p^2\gamma h}{gD}\frac{a^2}{10} = 0,$$

from which

$$p = \frac{10.33}{a^2}\sqrt{\frac{gD}{\gamma h}}. \tag{215}$$

In order to get a closer approximation we take the two first terms of the series (m), then

$$\int_0^a \left(\frac{d^2w_0}{dr^2} + \frac{1}{r}\frac{dw_0}{dr}\right)^2 rdr = \frac{96}{9a^2}\left(a_1^2 + \frac{3}{2}a_1a_2 + \frac{9}{10}a_2^2\right),$$

$$\int_0^a w_0^2 rdr = \frac{a^2}{10}\left(a_1^2 + \frac{5}{3}a_1a_2 + \frac{5}{7}a_2^2\right).$$

Equations (214) become

$$\begin{aligned} a_1\left(\frac{192}{9} - \frac{x}{5}\right) + a_2\left(\frac{144}{9} - \frac{x}{6}\right) &= 0, \\ a_1\left(\frac{144}{9} - \frac{x}{6}\right) + a_2\left(\frac{96}{5} - \frac{x}{7}\right) &= 0, \end{aligned} \tag{n}$$

where

$$x = a^4p^2\frac{\gamma h}{gD}. \tag{o}$$

Equating to zero the determinant of eqs. (n) we obtain

$$x^2 - \frac{204 \times 48}{5}x + 768 \times 36 \times 7 = 0,$$

from which

$$x_1 = 104.3; \qquad x_2 = 1854.$$

Substituting in (o) we obtain

$$p_1 = \frac{10.21}{a^2}\sqrt{\frac{gD}{\gamma h}}; \qquad p_2 = \frac{43.04}{a^2}\sqrt{\frac{gD}{\gamma h}}. \tag{216}$$

p_1 determines the second approximation to the frequency of the lowest mode of vibration of the plate and p_2 gives a rough approximation to the frequency of the second mode of vibration, in which the vibrating plate has one nodal circle. By using the same method the modes of vibration having nodal diameters can also be investigated.

In all cases the frequency of vibration will be determined by the equation

$$p = \frac{\alpha}{a^2}\sqrt{\frac{gD}{\gamma h}}, \tag{217}$$

the constant α of which for a given number s of nodal circles and of a given number n of nodal diameters is given in the table below.

s	$n = 0$	$n = 1$	$n = 2$
0	10.21	21.22	34.84
1	39.78	—	—
2	88.90	—	—

In the case of thin plates the mass of the air or of the liquid in which the plate vibrates may affect the frequency considerably. In order to take this into account in the case of the lowest mode of vibration, equation (217) above should be replaced by the following equation,*

$$p_1 = \frac{10.21}{a^2\sqrt{1+\beta}}\sqrt{\frac{gD}{\gamma h}}, \tag{218}$$

in which

$$\beta = .6689\,\frac{\gamma_1}{\gamma}\,\frac{a}{h}$$

and (γ_1/γ) = the ratio of the density of the fluid to the density of the material of the plate.

Taking, for instance, a steel plate of 7 inches diameter and 1/8 inch thick vibrating in water, we obtain

$$\beta = .6689 \times \frac{1}{7.8} \times 28 = 2.40; \qquad \frac{1}{\sqrt{1+\beta}} = .542.$$

The frequency of the lowest mode of vibration will be lowered to .542 of its original value.

Other Kinds of Boundary Conditions.—In all cases the frequencies of a vibrating circular plate can be calculated from eq. (217). The numerical values of the factor α are given in the tables below.

For a free circular plate with n nodal diameters and s nodal circles α has the following values: †

* This problem has been discussed by H. Lamb, Proc. Roy. Soc. London, Vol. 98 (1921), p. 205.

† Poisson's ratio is taken equal to 1/3.

s	$n = 0$	$n = 1$	$n = 2$	$n = 3$
0	—	—	5.251	12.23
1	9.076	20.52	35.24	52.91
2	38.52	59.86		

For a circular plate with its center fixed and having s nodal circles α has the following values *

$s =$	0	1	2	3
$\alpha =$	3.75	20.91	60.68	119.7

The frequencies of vibration having nodal diameters will be the same as in the case of a free plate.

The Effect of Stretching of the Middle Surface of the Plate.—In the previous theory it was assumed that the deflection of the plate is small in comparison with its thickness. If a vibrating plate is under considerable static pressure such that the deflection produced by this pressure is not small in comparison with the thickness of the plate, the stretching of the middle surface of the plate should be taken into account in calculating the frequency of vibration. Due to the resistance of the plate to such a stretching the rigidity of the plate and the frequency of vibration increase with the pressure acting on the plate.† In order to show how the stretching of the middle surface may affect the frequency, let us consider again the case of a circular plate clamped at the boundary and assume that the deflection of the plate under a uniformly distributed pressure is given by the equation ‡

$$w_0 = a_1\left(1 - \frac{r^2}{a^2}\right)^2. \qquad (m)^1$$

In addition to the displacements w_0 at right angles to the plate the points in the middle plane of the plate will perform radial displacements u which vanish at the center and at the clamped boundary of the plate. The unit elongation of the middle surface in a radial direction, due to

* See paper by R. V. Southwell, Proc. Roy. Soc., A, Vol. 101 (1922), p. 133; $\sigma = .3$ is taken in these calculations.

† Such an increase in frequency was established experimentally. See paper by J. H. Powell and J. H. T. Roberts, Proc. Phys. Soc. London, Vol. 35 (1923), p. 170.

‡ This equation represents the deflections when the stretching of the middle surface is neglected. It can be used also for approximate calculation of the effect of the stretching.

the displacements w_0 and u, is

$$e_r = \frac{1}{2}\left(\frac{dw_0}{dr}\right)^2 + \frac{du}{dr}. \tag{p}$$

The elongation in a circumferential direction will be,

$$e_t = \frac{u}{r}. \tag{r}$$

For an approximate solution of the problem we assume that the radial displacements are represented by the following series:

$$u = r(a - r)(c_1 + c_2 r + c_3 r^2 + \cdots), \tag{s}$$

each term of which satisfies the boundary conditions.

Taking only the first two terms in the series (s) and substituting (s) and (m)[1] in eqs. (p) and (r) the strain in the middle surface will be obtained and the energy corresponding to the stretching of the middle surface can now be calculated as follows:

$$V_1 = \frac{\pi Eh}{1-\sigma^2}\int_0^a (e_r^2 + e_t^2 + 2\sigma e_r e_t) r dr = \frac{\pi Eha^2}{1-\sigma^2}\Big(.250c_1^2a^2 + .1167c_2^2a^4 + .300c_1c_2a^3 - .00846c_1a\frac{8a_1^2}{a^2} + .00682c_2a^2\frac{8a_1^2}{a^2} + .00477\frac{64a_1^4}{a^4}\Big). \tag{t}$$

Determining the constants c_1 and c_2 so as to make V_1 a minimum, we get, from the equations

$$\frac{\partial V_1}{\partial c_1} = 0, \qquad \frac{\partial V_1}{\partial c_2} = 0,$$

$$c_1 = 1.185\frac{a_1^2}{a^3}; \qquad c_2 = -1.75\frac{a_1^2}{a^4}.$$

Substituting in eq. (t):

$$V_1 = 2.59\pi D\frac{a_1^4}{a^2h^2}.$$

Adding this energy of stretching to the energy of bending (eq. 211) we obtain,

$$V = \frac{32}{3}\pi D\frac{a_1^2}{a^2} + 2.59\pi D\frac{a_1^4}{a^2h^2} = \frac{32}{3}\pi D\frac{a_1^2}{a^2}\left(1 + .244\frac{a_1^2}{h^2}\right). \tag{u}$$

The second term in the brackets represents the correction due to the extension of the middle surface of the plate. It is easy to see that this correction is small and can be neglected only when the deflection a_1 at the center of the plate is small in comparison with the thickness h. The static deflection of the plate under the action of a uniformly distributed pressure q can now be found from the equation of virtual displacements,

$$\frac{\partial V}{\partial a_1}\delta a_1 = 2\pi q \delta a_1 \int_0^a \left(1 - \frac{r^2}{a^2}\right)^2 r dr = \delta a_1 \frac{\pi q a^2}{3}$$

from which

$$a_1 = \frac{qa^4}{64D} \cdot \frac{1}{1 + .488\dfrac{a_1^2}{h^2}} \cdot \tag{219}$$

The last factor on the right side represents the effect of the stretching of the middle surface. Due to this effect the deflection a_1 is no longer proportional to q and the rigidity of the plate increases with the deflection. Taking, for instance, $a_1 = \frac{1}{2}h$, we obtain, from (219)

$$a_1 = .89\frac{qa^4}{64D} \cdot$$

The deflection is 11% less than that obtained by neglecting the stretching of the middle surface.

From the expression (u) of the potential energy, which contains not only the square but also the fourth power of the deflection a_1, it can be concluded at once that the vibration of the plate about its flat configuration will not be *isochronic* and the frequency will increase with the amplitude of vibration. Consider now small vibrations of the plate about a bent position given by eq. $(m)^1$. This bending is supposed to be due to some constant uniformly distributed static pressure q. If Δ denotes the amplitude of this vibration, the increase in the potential energy of deformation due to additional deflection of the plate will be obtained from eq. (u) and is equal to *

$$\delta V = \frac{32}{3}\frac{\pi D}{a^2}\left(2a_1\Delta + \Delta^2 + \frac{.244}{h^2}(4a_1^3\Delta + 6a_1^2\Delta^2)\right) \cdot$$

The work done by the constant pressure q during this increase in deflection is

$$\delta W = \frac{\pi a^2 q \Delta}{3} = \frac{\pi a^2 \Delta}{3} \cdot \frac{64a_1 D\left(1 + .488\dfrac{a_1^2}{h^2}\right)}{a^4} \cdot$$

* Terms with Δ^3 and Δ^4 are neglected in this expression.

The complete change in the potential energy of the system will be

$$\delta V - \delta W = \frac{32}{3}\frac{\pi D \Delta^2}{a^2}\left(1 + \frac{1.464 a_1^2}{h^2}\right).$$

Equating this to the maximum kinetic energy,

$$T_{\max} = \frac{\pi \Delta^2 p^2 \gamma h}{g}\int_0^a \left(1 - \frac{r^2}{a^2}\right)^4 r dr = \frac{\pi \Delta^2 a^2 \gamma h}{10 g} p^2$$

we obtain

$$p = \frac{10.33}{a^2}\sqrt{\frac{gD}{\gamma h}}\sqrt{1 + 1.464\frac{a_1^2}{h^2}}. \tag{220}$$

Comparing this result with equation (215) it can be concluded that the last factor on the right side of eq. (220) represents the correction due to the stretching of the middle surface of the plate.

It should be noted that in the above theory equation $(m)^1$ for the deflection of the plate was used and the effect of tension in the middle surface of the plate on the form of the deflection surface was neglected. This is the reason why eq. (220) will be accurate enough only if the deflections are not large, say $a_1 \leq h$. Otherwise the effect of tension in the middle surface on the form of the deflection surface must be taken into consideration.

57. Vibration of Turbine Discs.—*General.*—It is now fairly well established that fractures which occur in turbine discs and which cannot be explained by defects in the material or by excessive stresses due to centrifugal forces may be attributed to flexural vibrations of these discs. In this respect it may be noted that direct experiments have shown * that such vibrations, at certain speeds of the turbine, become very pronounced and produce considerable additional bending stresses which may result in fatigue of the metal and in the gradual development of cracks, which usually start at the boundaries of the steam balance holes and other discontinuities in the web of the turbine disc, where stress concentration is present.

There are various causes which may produce these flexural vibrations in turbine discs but the most important is that due to non-uniform steam pressure. A localized pressure acting on the rim of a rotating disc is sufficient at certain speeds to maintain lateral vibrations in the disc and experiments show that the application of a localized force of only a few

* See paper by Wilfred Campbell, Trans. Am. Soc. Mech. Eng., Vol. 46 (1924) p. 31. See also paper by Dr. J. von Freudenreich, Engineering, Vol. 119 p. 2, (1925).

pounds, such as produced by a small direct current magnet to the side of a rotating turbine disc makes it respond violently at a whole series of critical speeds.

Assume now that there exists a certain irregularity in the nozzles which results in a non-uniform steam pressure and imagine that a turbine disc is rotating with a constant angular velocity ω in the field of such a pressure. Then for a certain spot on the rim of the disc the pressure will vary with the angle of the rotation of the wheel and this may be represented by a periodic function, the period of which is equal to the time of one revolution of the disc. In the most general case such a function may be represented by a trigonometrical series

$$q = a_0 + a_1 \sin \omega t + a_2 \sin 2\omega t + \cdots b_1 \cos \omega t + b_2 \cos 2\omega t + \cdots.$$

By taking only one term of the series such as $a_1 \sin \omega t$ we obtain a periodic disturbing force which may produce large lateral vibration of the disc if the frequency $\omega/2\pi$ of the force coincides with one of the natural frequencies $p/2\pi$ of the disc. From this it can be appreciated that the calculation of the natural frequencies of a disc may have a great practical importance.

A rotating disc, like a circular plate, may have various modes of vibration which can be sub-divided into two classes:

a. Vibrations symmetrical with respect to the center, having nodal lines in the form of concentric circles, and

b. Unsymmetrical having diameters for nodal lines. The experiments show that the symmetrical type of vibration very seldom occurs and no disc failure can be attributed to this kind of vibration.

In discussing the unsymmetrical vibrations it can be assumed that the deflection of the disc has the following form,

$$w = w_0 \sin n\theta \cos pt, \qquad (a)$$

in which, as before, w_0 is a function of the radial distance r only, θ determines the angular position of the point under consideration, and n represents the number of nodal diameters.

The deflection can be taken also in the form

$$w = w_0 \cos n\theta \sin pt. \qquad (a)^1$$

Combining (a) and $(a)^1$ we obtain

$$w = w_0(\sin n\theta \cos pt \pm \cos n\theta \sin pt) = w_0 \sin (n\theta \pm pt),$$

which represents traveling waves. The angular speed of these waves

traveling around the disc will be found from the condition

$$n\theta \pm pt = \text{const.},$$

from

$$\theta = \pm \frac{p}{n} t + \text{const.}$$

we obtain two speeds $-p/n$ and $+p/n$ which are the speeds of the backward and forward traveling waves, respectively. The experiments of Campbell * proved the existence of these two trains of waves in a rotating disc and showed also that the amplitudes of the backward moving waves are usually larger than those of the forward moving waves. Backward moving waves become especially pronounced under conditions of resonance when the backward speed of these waves in the disc coincides exactly with the forward angular velocity of the rotating disc so that the waves become *stationary in space.* The experiments show that this type of vibration is responsible in a majority of cases for disc failures.

Calculation of the Frequencies of Disc Vibrations.—In calculating the frequencies of the various modes of vibration of turbine discs the Rayleigh-Ritz method is very useful.† In applying this method we assume that the deflection of the disc has a form

$$w = w_0 \sin n\theta \cos pt. \qquad (a)''$$

In the particular case of vibration symmetrical with respect to the center the deflection will be:

$$w = w_0 \cos pt. \qquad (b)$$

Considering in the following this particular case the maximum potential energy of deformation will be, from eq. (209),

$$V_{\max} = \pi \int_b^a D \left\{ \left(\frac{d^2 w_0}{dr^2} + \frac{1}{r}\frac{dw}{dr} \right)^2 - 2(1-\sigma) \frac{d^2 w}{dr^2} \frac{1}{r} \frac{dw}{dr} \right\} r dr, \qquad (c)$$

where a, b = outer and inner radii of the disc,

$$D = \frac{Eh^3}{12(1-\sigma^2)} = \text{flexural ridigity of the disc, which in this case}$$

will be variable due to variation in thickness h of the disc.

In considering the vibration of a rotating disc not only the energy of deformation but also the energy corresponding to the work done during

* Loc. cit., p. 320.

† The vibration of turbine discs by using this method was investigated by A. Stodola, Schweiz. Bauz., Vol. 63, p. 112 (1914).

deflection by the centrifugal forces must be taken into consideration. It is easy to see that the centrifugal forces resist any deflection of the disc and this results in an increase in the frequency of its natural vibration. In calculating the work done by the centrifugal forces let us take an element cut out from the disc by two cylindrical surfaces of the radii r and $r + dr$ (Fig. 143). The radial displacement of this element towards the center due to the deflection will be

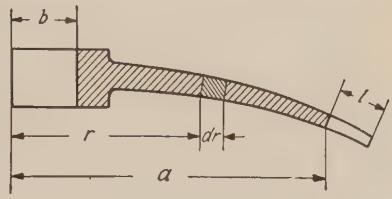


FIG. 143.

$$\frac{1}{2}\int_b^r \left(\frac{dw_0}{dr}\right)^2 dr.$$

The mass of the element is

$$\frac{2\pi r h\gamma}{g}\, dr$$

and the work done during the deflection by the centrifugal forces acting on this element will be

$$-\frac{2\pi r^2\omega^2 h\gamma}{g}\, dr \cdot \frac{1}{2}\int_b^r \left(\frac{dw_0}{dr}\right)^2 dr. \qquad (d)$$

The energy corresponding to the work of the centrifugal forces will be obtained by summation of such elements as (d) in the following form,

$$V_{1\max} = \int_b^a \frac{\pi r^2\omega^2 h\gamma}{g}\, dr \int_b^r \left(\frac{dw_0}{dr}\right)^2 dr. \qquad (e)$$

The maximum kinetic energy is given by the equation

$$T = \int_b^a \frac{2\pi r\gamma h}{2g}\, \dot{w}^2 dr.$$

Substituting expression (b) for w we obtain

$$T_{\max} = \frac{\pi\gamma p^2}{g}\int_b^a h w_0^2 r dr. \qquad (f)$$

Now, from the equation

$$V_{\max} + V_{1\max} = T_{\max}$$

we deduce

$$p^2 = \frac{V_{\max} + V_{1\max}}{\frac{\pi\gamma}{g}\int_b^a hw_0^2 r dr}. \tag{g}$$

In order to obtain the frequency the deflection curve w_0 should be chosen so as to make the expression (g) a minimum. This can be done graphically by assuming for w_0 a suitable curve from which w_0, dw_0/dr and d^2w_0/dr^2 can be taken for a series of equidistant points and then the expressions (c), (e) and (f) can be calculated. By gradual changes in the shape of the curve for w_0 a satisfactory approximation for the lowest frequency can be obtained * from eq. (g).

In order to take into account the effect of the blades on the frequency of natural vibration the integration in the expression (e) and (f) for the work done by the centrifugal forces and for the kinetic energy must be extended from b to $a + l$ where l denotes the length of the blade. In this calculation the blades can be assumed to be straight during vibration of the disc so that no addition to the expression for the potential energy (c) will be necessary.

In an analytical calculation of the lowest frequency of a vibrating disc we take w_0 in the form of a series such as

$$w_0 = a_1(r - b)^2 + a_2(r - b)^3 + a_3(r - b)^4 + \cdots,$$

which satisfies the conditions at the built-in inner boundary of the disc, where w_0 and dw_0/dr become equal to zero. The coefficients a_1, a_2, a_3 should now be chosen so as to make expression (g) a minimum. Proceeding as explained in the previous paragraph (see p. 314) a system of equations analogous to the equations (214) and linear in a_1, a_2, $a_3 \cdots$ can be obtained. Equating to zero the determinant of these equations, the frequency equation will be found.

In the case of a mode of vibration having diameters as nodal lines the expression $(a)''$ instead of (b) must be used for the deflections. The potential energy will be found from eq. (208): it is only necessary to take into consideration that in the case of turbine discs the thickness and the flexural rigidity D are varying with the radial distance r so that D must be retained under the sign of integration. Without any difficulty also the expressions for V_1 and T can be established for this case and

* Such a graphical method has been developed by A. Stodola, loc. cit., p. 66. It was applied also by E. Oehler, V. D. I., Vol. 69 (1925), p. 335 and gave good agreement with experimental data.

finally the frequency can be calculated from eq. (g) exactly in the same manner as it was explained above for the case of a symmetrical mode of vibration.*

When the disc is stationary V_1 vanishes and we obtain from equation (g)

$$p_1^2 = \frac{V_{\max}}{\frac{\pi\gamma}{g}\int_b^a hw_0^2 r dr}, \qquad (g)'$$

which determines the frequency of vibration due to elastic forces alone.

Another extreme case is obtained when the disc is very flexible and the restoring forces during vibration are due entirely to centrifugal forces. Such conditions are encountered, for instance, when experimenting with flexible discs made of rubber. The frequency will be determined in this case from eq.

$$p_2^2 = \frac{V_{1\max}}{\frac{\pi\gamma}{g}\int_b^a hw_0^2 r dr}. \qquad (g)''$$

Now, from eq. (g), we have

$$p^2 = p_1^2 + p_2^2. \qquad (h)$$

If the frequencies p_1 and p_2 are determined in some way, the resulting frequency of vibration of the disc will be found from eq. (h). In the case of discs of constant thickness and fixed at the center an exact solution for p_1 and p_2 has been obtained by R. V. Southwell.† He gives for p_1^2 the following equation,

$$p_1^2 = \frac{\alpha}{a^4}\frac{Dg}{\gamma h}. \qquad (k)$$

The values of the constant α for a given number n of nodal diameters and a given number s of nodal circles are given in the table below.‡

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
$s = 0$	14.1	0	29.0	156
$s = 1$	438	422	1210	2840

* The formulae for this calculation are developed in detail by A. Stodola, loc. cit.

† Loc. cit., p. 317.

‡ All other notations are the same as for circular plates (see p. 314). Poisson's ratio is taken equal to .3 in these calculations.

The equation for calculating p_2^2 is

$$p_2^2 = \lambda\omega^2, \tag{l}$$

in which ω is the angular velocity and λ is a constant given in the table below,

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
$s = 0$	0	1	2.35	4.05
$s = 1$	3.3	5.95	8.95	12.3

Determining p_1^2 and p_2^2 from the equations (k) and (l) the frequency of vibration of the rotating disc will then be found from eq. (h).*

In the above theory of the vibration of discs the effect of non-uniform heating of the disc was not considered. In a turbine in service the rim of the disc will be warmer than the web. Due to this factor compressive stresses in the rim and tensile stresses in the web will be set up which may affect the frequencies of the natural vibrations considerably. The experiments and calculations † show that for vibrations with 0 and 1 nodal diameters, the frequency is increased, whereas with a larger number of nodal diameters, the frequency is lowered by such a non-uniform heating.

* A discussion of the differential equation of vibration for the case of a disc of variable thickness is given in the paper by Dr. Fr. Dubois, Schweiz. Bauz., Vol. 89, p. 149 (1927).

† Freudenreich, loc. cit., p. 320.

APPENDIX

VIBRATION MEASURING INSTRUMENTS

1. General.—Until quite recently practical vibration problems in the shops and in the field were usually left to the care of men who did not have great knowledge of the theory of vibration and based their opinions on data obtained from experience and gathered by the unaided senses of touch, sight and hearing. With the increasing dimensions and velocities of modern rotating machinery, the problem of eliminating vibrations becomes more and more important and for a successful solution of this problem the compilation of *quantitative* data on the vibrations of such machines and their foundations becomes necessary. Such quantitative results, however, can be got only by means of *instruments*. The fundamental data to be measured in investigating this problem are: (*a*) the frequency of the vibration, (*b*) its amplitude, (*c*) the type of wave, simple harmonic, or complex, and (*d*) the stresses produced by this vibration.

Modern industry developed many instruments for measuring the above quantities and in the following some of the most important, which have found wide application, will be described.*

2. Frequency Measuring Instruments.—A knowledge of the frequency of a vibration is very important and often gives a valuable clue to its source. The description of a very simple frequency meter, *Frahm's tachometer*, which has long been used in turbo generators, was given before. (See page 15.) The *Fullarton vibrometer* is built on the same principle. It is shown in Fig. 144. This instrument consists of a claw *A* to be clamped under a bolt head, two joints *B* rotatable at right angles to each other, a main frame bearing a reed *C*, a length scale *D* on the side, an amplitude scale *E* across the top, and a long screw *F*. A clamp carriage rides on the main frame, its position being adjusted by the screw. The reed is held tightly in a fixed clamp at the bottom of the frame and its free length is varied by the position of the movable clamp

* See the paper by J. Ormondroyd, Journal A. I. E. E., Vol. 45 (1926), p. 330. See also the paper by P. A. Borden, A. I. E. E. Trans., 1925, p. 238, and the paper by H. Steuding, V. D. I., Vol. 71 (1927), p. 605, representing an abstract from a very complete investigation on vibration recording instruments made for the Special Committee on Vibration organized by the V. D. I. (Society of German Engineers).

on the carriage. The instrument is bolted to the vibrating machine * and the free length of the reed is adjusted until the largest amplitude of motion is obtained at the end of the reed. This is read on the transverse

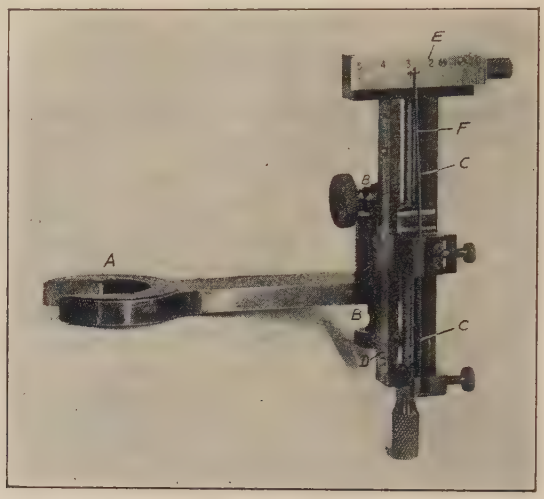


Fig. 144.

scale. The instrument then is in resonance with the impressed frequency. This frequency can be determined by measuring the free length of the reed.

This device is so highly selective (damping forces extremely small) that it can be used only on vibrations with almost absolutely constant frequency. The least variation in frequency near the resonance point will give a very large fluctuation in amplitude. This limits the instrument to uses on turbo generators and other machinery in which the speed varies only slightly.

3. The Measurement of Amplitudes.—There are many instances where it is important to measure only the amplitude of the vibration. This is true in most cases of studying forced periodic vibrations of a known frequency such as are found in structures or apparatus under the action of rotating machinery. Probably the most frequent need for measuring amplitudes occurs in power plants, where vibrations of the building, of the floor, of the foundation or of the frame are produced by impulses given once, a revolution due to unbalance of the rotating parts.

* The weight of the machine should be considerably larger than the weight of the instrument to exclude the possibility of the instrument affecting the motion of the machine.

The theory on which seismic instruments are based is given on page 14. An amplitude meter on this principle, built by the Vibration Specialty Company of Philadelphia, is shown in Figure 145. The photograph shows the instrument with the side cover off. It is of the seismographic

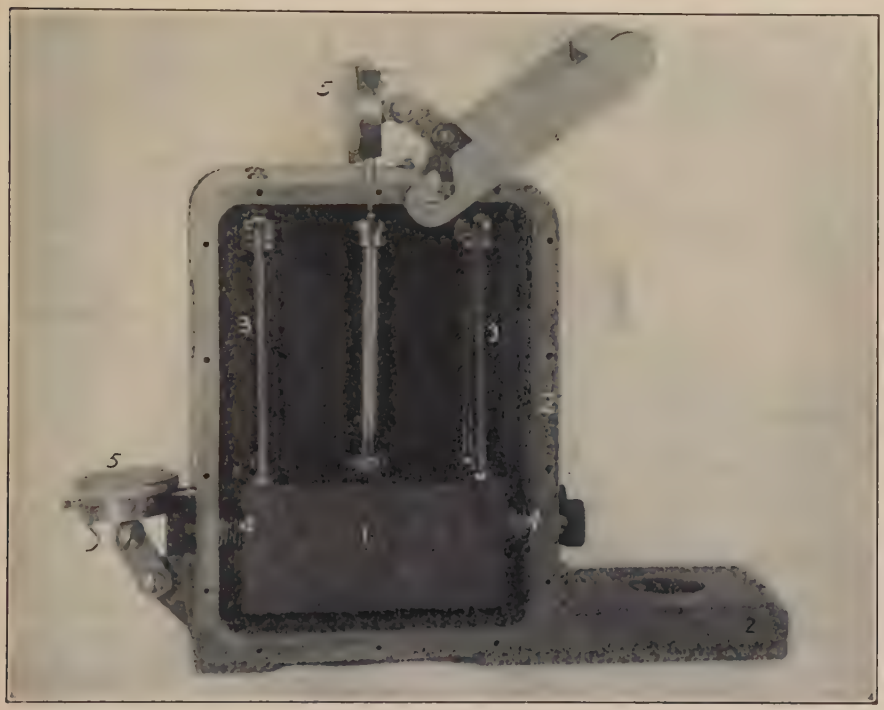


Fig. 145.

type. A steel block (1) is suspended on springs (3) in a heavy frame (2), the additional compression springs (4) centering the block horizontally. The frequencies of the natural vibrations of the block both in vertical and horizontal direction are about 200 per minute. The frame carries two dial indicators (5), the plungers of which touch the block. The instrument is to be bolted to the structure under investigation. The frequency of vibration produced by high speed rotating machinery is usually several times higher than the natural frequency of the vibrometer and the block of the instrument can be considered as stationary in space. The indicators register the vertical and horizontal components of the

relative motion between the block and the frame, their hands moving back and forth over arcs giving the double amplitudes of these components.

This instrument proved to be very useful in power plants for studying the vibration of turbo generators. It is a well known fact that at times a unit, due probably to non-uniform temperature distribution in the rotor, begins to vibrate badly when brought to full speed, the vibrations persisting for a long period. This condition may be cured by slowing the machine down and then raising the speed again. Sometimes vibrations may be built up also at changes in the load or due to a drop in the vacuum, which is accompanied by variations in temperature of the turbine parts. One or two vibrometers mounted on the bearing pedestals of the turbine will give complete information about such vibrations.

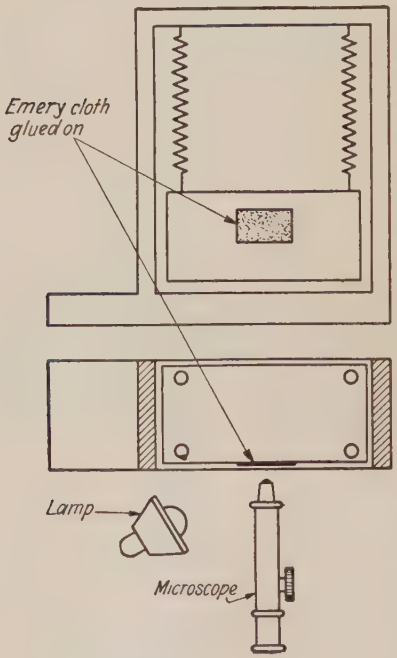


Fig. 146.

The instrument is also very useful for balancing the rotors at high speed, especially when a very fine balancing is needed. The elimination of the personal element during this operation is of great importance. The balancing takes a long time when the unit is in service, several days passing sometimes between two consecutive trials and a numerical record of the amplitude of vibration gives a definite method of comparing the condition of the machine for the various locations of balancing weights. The procedure of balancing by using only the amplitudes of the vibration was described before (see page 46).

Another interesting application of this instrument is shown in Fig. 146. With the front cover off the instrument, the actual path of a point on the vibrating pedestal of a turbo generator can be studied.* A piece of emery cloth of a medium grade is glued to the steel block of the instrument. A light is thrown onto the emery, giving very sharp point

* This method was devised by Mr. G. B. Karelitz, Research Engineer of the Westinghouse Electric & Manufacturing Company.

reflections on the crystals of carborundum. A microscope is rigidly attached to the pedestal under investigation and focused on the emery cloth. The block being stationary in space, the relative motion of the microscope and the cloth can be clearly seen, the points of light scribing

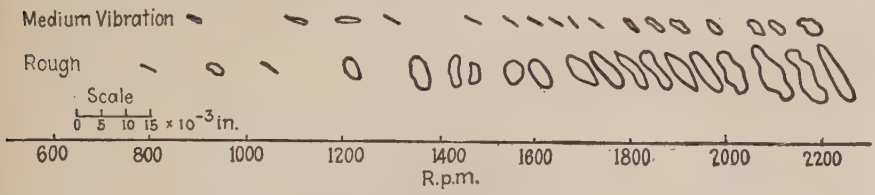


Fig. 147.

bright figures, of the same kind as the well known Lissajous' figures. Typical figures as obtained on a pedestal of an 1800 r.p.m. turbine are shown in Fig. 147.

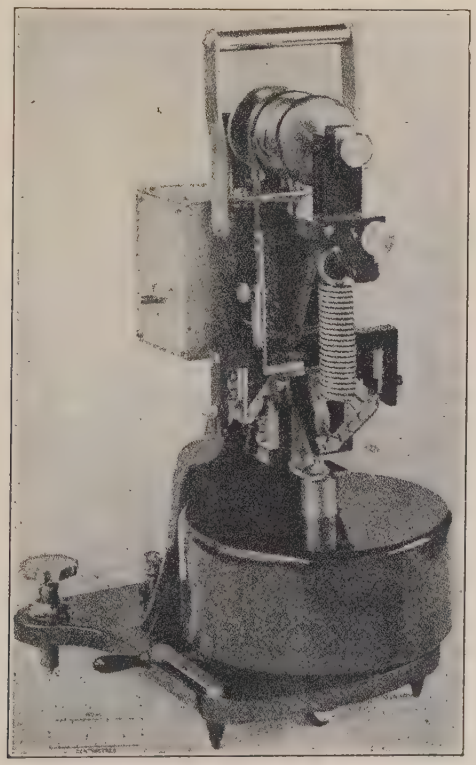


Fig. 148.

4. Seismic Vibrographs.—Seismic vibrographs are used where a complete analysis of the vibration is required. The chief application these instruments find is in the measurement of floor vibrations in buildings, vibrations of foundations of machines and vibrations of bridges. By analyzing a vibrograph record into simple harmonic vibrations, it is possible sometimes to find out the source of the disturbing forces producing these component vibrations.

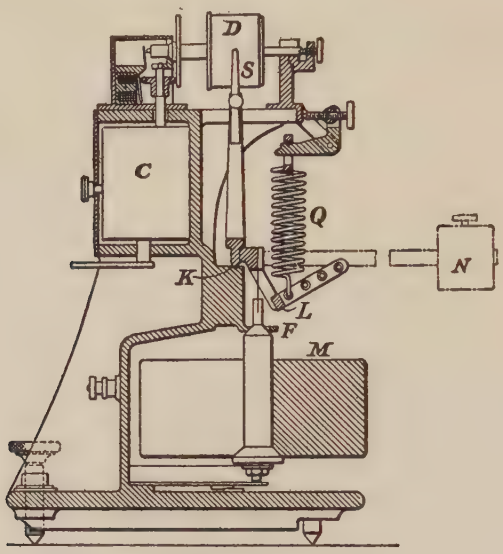


Fig. 149.

The Vibrograph constructed by the Cambridge Instrument Company * is shown in Figs. 148 and 149. This instrument records vertical vibrations. If required for violent oscillations, the instrument is fitted with a steel yard attachment indicated by the dotted lines of the sectional diagram, Fig. 149. The instrument consists of a weighted lever, pivoted on knife edges on a stand which, when placed on the structure or foundation, partakes of its vibrations. The small lever movements caused by the vibrations are recorded on a moving strip of celluloid by a fine point carried at the extremity of an arm joined to the lever. The heavy mass M is attached by a metal strip to a steel block which is pivoted to the stand by means of the knife edges K. The steel block forms a short

* For a more detailed description of this instrument, see Engineering, Vol. 119 (1925), p. 271.

lever, the effective length of which is equal to the horizontal distance between the strip and the knife edges. The weight M is balanced by a helical spring Q suspended from the upper portion of the stand. The lower end of this spring is hooked into one of the four holes in the arm of the bell crank lever L and by selecting one of these holes the natural frequency of the moving system can be altered. An arm extending upward from the pivoted steel block, previously referred to, has at its upper extremity a flat spring S, carrying the recording point. This point bears upon the surface of a celluloid film (actually a portion of clear moving picture film) wrapped around the split drum D which is rotated by means of the clockwork mechanism C. By means of an adjustable governor the speed of the film can be varied between about 4 mm. and 20 mm. per second. In the narrow gap between the two portions of the split drum D rests a second point which can be shifted laterally by means of an electromagnet acting through a small lever mechanism inside the drum. This electromagnet is connected to a separate clock, making contact every tenth of a second, or other time interval. Thus a zero line with time markings is recorded on the back of the film simultaneously with the actual "vibrogram" on the front. The records obtained can be read by a microscope accurately to .01 mm. and as the initial magnification of the recording instrument is 10, a vertical movement of the foundation of 10^{-4} cm. is clearly measurable.* In Fig. 150, the "Geiger" Vibrograph is shown.† The whole instrument, the dimensions of which are about 8″ x 6″ x 6″, has to be attached to the vibrating machine or structure. A heavy block on weak springs supported inside the instrument will remain still in space. The relative motion between this block and the frame of the instrument is transmitted to a capillary pen which traces a record of it on a band of paper, 2½″ wide. A clockwork, which can be set at various speeds, moves the band of paper and rolls it up on a pulley. For time marking there is a cantilever spring attached to the frame with a steel knob and a pen on its end. This cantilever has a natural frequency of 25 cycles per second. It can be operated either by hand or electrically by means of two dry cells. It must be deflected every second or so and traces a damped 25-cycle wave on the record. The natural period of the seismographic mass

* This method of recording was first adopted by W. G. Collins in the Cambridge microindicator for high-speed engines, see Engineering, Vol. 13 (1922), p. 716. See also Trans. of the Optical Society, Vol. 27 (1925–1926), p. 215.

† For a more detailed description of this instrument, see V. D. I., Vol. 60 (1916), p. 811.

itself is approximately 1½ per second. The magnification of the lever system connecting this mass with the pen is adjustable. Satisfactory records can be obtained with a magnification of 12 times for frequencies up to 130 per second. It will operate satisfactorily even to 200 cycles per second, provided the magnification chosen is not more than three times. It should be noted that by means of an adjustment at the seismographic mass it is possible to obtain a record of the vibration *in any direction.*

In cases where the vibrating body is so small that its vibration will

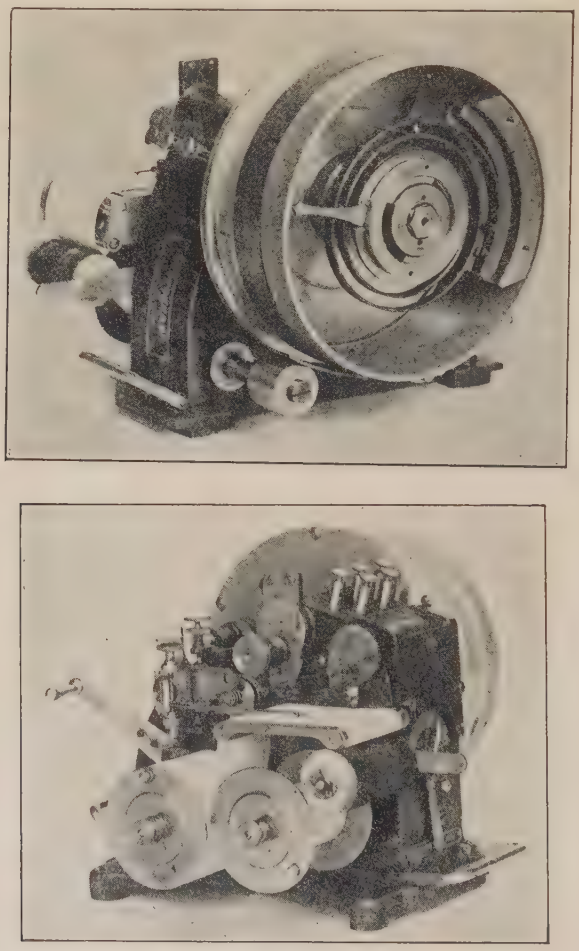


FIG. 150.

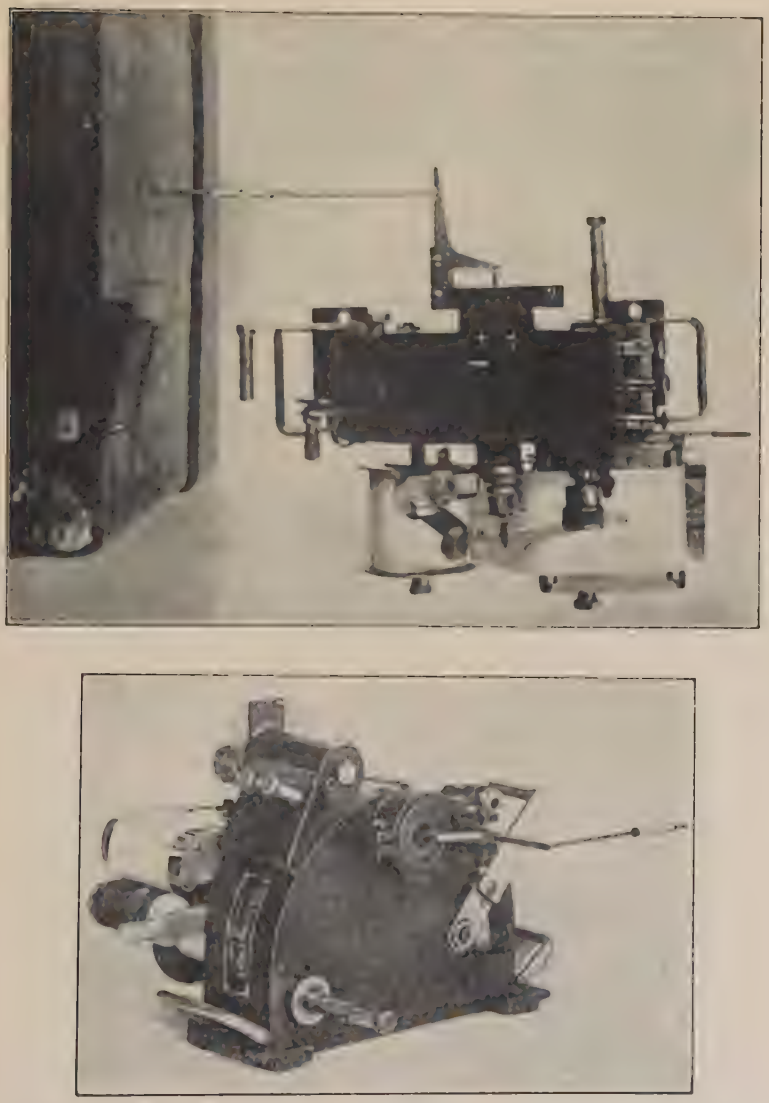


Fig. 151.

be affected by the comparatively large mass of the instrument, it is possible to use it merely as a recorder ("universal recorder," as it is called by the inventor). The seismographic mass is then taken out of it and the instrument has to be supported immovable in space in some

manner; for instance, by suspending it from a crane. The lever system of the recording pen is directly actuated by a tiny rod which touches the vibrating body. (Fig. 151). With this arrangement, magnifications of 100 times at 60 cycles and of 15 times at 150 cycles can be obtained. A record of this instrument is shown in Fig. 152.

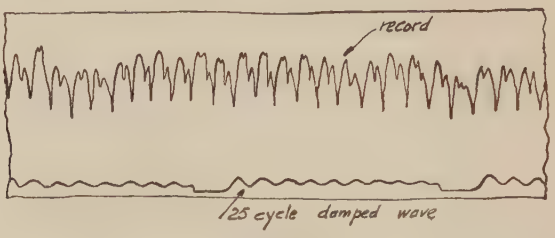


Fig. 152.

5. Torsiograph.—Many instruments have been designed for recording torsional vibrations in shafting. An instrument of this kind which has found a large application is shown in Fig. 153. This instrument,

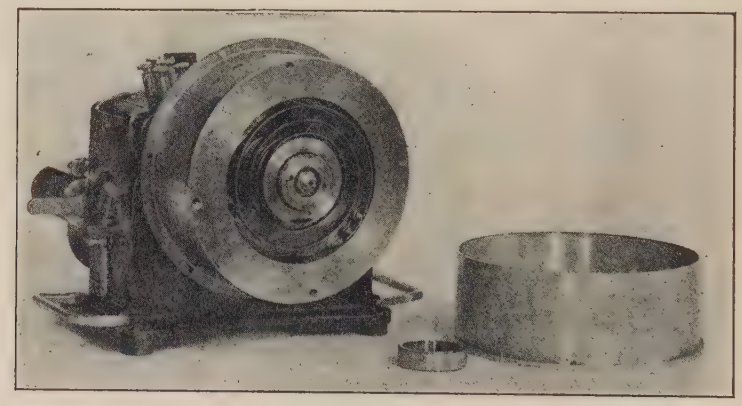


Fig. 153.

designed by A. Geiger, has the same recording and timing device as the vibrograph described above, but differs from it in its seismographic part. It has a light pulley of about 6″ diameter, in which a heavy fly-wheel is mounted concentrically and free to turn on the same axis. The connection between pulley and mass is by means of a very flexible spiral spring. The natural frequency of torsional oscillations of this mass, when the pulley is kept steady, is about 1½ per second. In

operation the pulley is driven by means of a short belt, 1″ wide, from the shaft of which the torsional oscillations are to be measured. The pulley moves with the shaft, but the heavy mass inside will revolve at practically uniform angular velocity provided that the frequency of torsional vibrations is above a certain value, say four times larger than the natural frequency of the instrument. The relative motion of the pulley and a point on the circumference of the fly-wheel is transmitted through a lever system to the recording pen. This instrument operates up to 200 cycles per second for low magnifications, and the magnification of the oscillatory motion on the circumference of the shaft can be made as high as 24 to 1 for low frequency motions. Small oscillations should be recorded from a portion of the shaft with as large a diameter as possible. Large oscillations should be measured on small diameter shafts to keep the record within the limits of the instrument. The limit to the size of the driving pulley is established by the effects of centrifugal forces on the spiral spring which is attached between the fly wheel and the pulley. At about 1500 r.p.m. the centrifugal forces distort the spring enough to push the pen off the recording strip. This instrument has been succesfully applied in studying torsional vibrations in Diesel engine installations such as in locomotives and submarines. Recently a combined torsiograph—vibrograph—universal recorder has been put on the market.

6. Torsion Meters.—There are cases where not only the oscillations of angular velocity as measured by Geiger's Torsiograph, but also the torque in a shaft transmitting power, is of interest. Many instruments have been designed for this purpose, especially in connection with measuring the power transmitted through propeller shafts of ships.* The generally accepted method is to measure the relative movement of two members fixed in two sections at a certain distance from each other on the shaft. The angle made by these members relative to each other is observed or recorded by an oscillograph. Knowing the speed of rotation of the shaft and its modulus of rigidity, the horse power transmitted can be determined. Fig. 154 represents the torsion meter designed by E. B. Moullin of the Engineering Laboratory, Cambridge, England.† "The relative movement of the two members of the instrument is

* There are various methods of measuring and recording the angle of twist in shafts, to be divided in four groups: (*a*) mechanical, (*b*) optical, (*c*) stroboscopic, and (*d*) electrical methods. Descriptions of the instruments built on these various principles are given in the paper by H. Steuding, mentioned above. (See page 327). See also the paper by V. Vieweg in the periodical "Der Betrieb," 1921, p. 378.

† See the paper by Robert S. Whipple, Journal of the Optical Soc. of America, Vol. 10 (1925), p. 455.

measured electrically and continuously throughout the revolution, so that the instrument can be fixed in the ship's tunnel on the shaft, and the observations made at a distance. The Moullin torsion meter has been

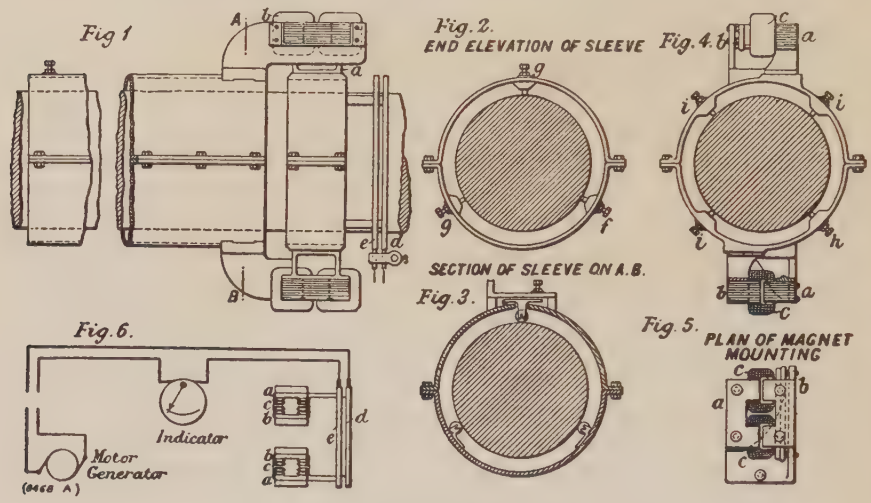


Fig. 154.

used to measure the torque transmitted on ships' shafts up to 10 inches in diameter, and transmitting 1500 H.P. The instrument consists of an air-gap choker, one-half carried by a ring fixed to a point on the shaft, and the other half carried adjacent to the first but attached to a sleeve fixed to the shaft about four feet away. Fig. 154 shows the arrangement of the halves of the choker, of which the one *a* is fixed to the ring, and the other *b* is attached to the sleeve. A small alternating current generator supplies current to the windings *c* at a frequency of 60 cycles per second and about 100 volts. As the shaft twists, the gap opens for forward running (and closes in running astern) and the current increases in direct proportion to the gap so that the measurements on a record vary directly with the torque. Two chokers are fitted, one at each end of a diameter, so that they are in mechanical balance, and, being connected electrically in series, are unaffected by bending movements. Current is led in and out of the chokers by two slip rings *d* and *e*." By using a standard oscillograph a continuous record can be obtained such as shown in Fig. 155.

In Fig. 156 is shown the torsion meter of Amsler, which is largely used for measuring the efficiency of high speed engines.

The connecting flanges D and L of the torsion meter are usually keyed on to the ends 1 and 2 of the driving and the driven shafts. The elastic bar which transmits the torsional effort is marked G. It is fitted

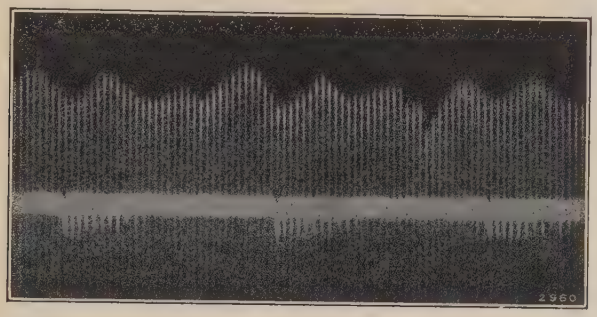


Fig. 155.

at the ends in the chucks F and H. The chuck F is always fastened to the sleeve A on which the flange B is keyed. The flange B is bolted to the flange D, and the flange J to the flange L; the ends of the bar G are

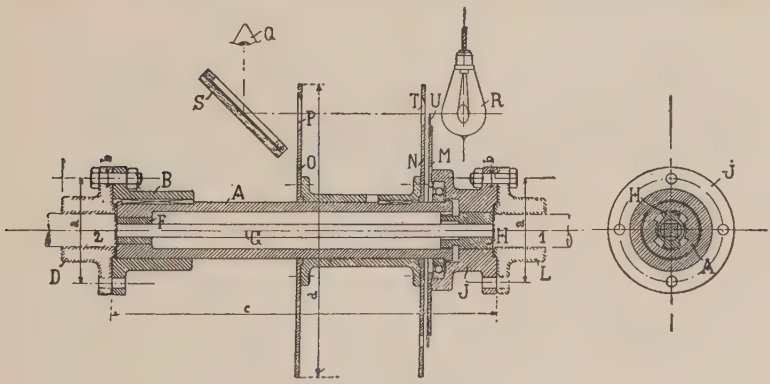


Fig. 156.

thus rigidly secured to the flanges D and L. In order to measure the angle of twist the discs M, N and O are used. M is fastened to the chuck J, while the other two, N and O, are fixed to the sleeve A. When the measuring bar G is twisted under the action of a torque, the disc M turns with respect to the other two discs N and O through a definite angle of twist. The edge U of the disc M is made of a ring of transparent celluloid on which a scale is engraved. Opposite this scale there is a small opening T in the disc N, and a fine radial slot which serves as a

pointer for making readings on the scale. The disc O has no opening opposite T but only a radial slot like the one in the disc N, and through this the observer looks when reading the position of the indicator T on the scale U by means of the mirror S placed at an angle of 45 degrees to the visual ray. The scale engraved on the celluloid is well illuminated from behind by means of a lamp R. If the apparatus has a considerable velocity, say not less than 250 revolutions per minute, the number of luminous impressions per second will be sufficient to give the impression of a steady image and the reading of the angle of twist can be taken with a great accuracy, provided this angle remains constant during rotation. Knowing the angle of twist and the torsional rigidity of the bar G, the torque moment and the power transmitted can easily be calculated.

V. Vieweg improved the instrument described above by attaching the mirror S to the disc as shown in Fig. 157 and by taking the distance

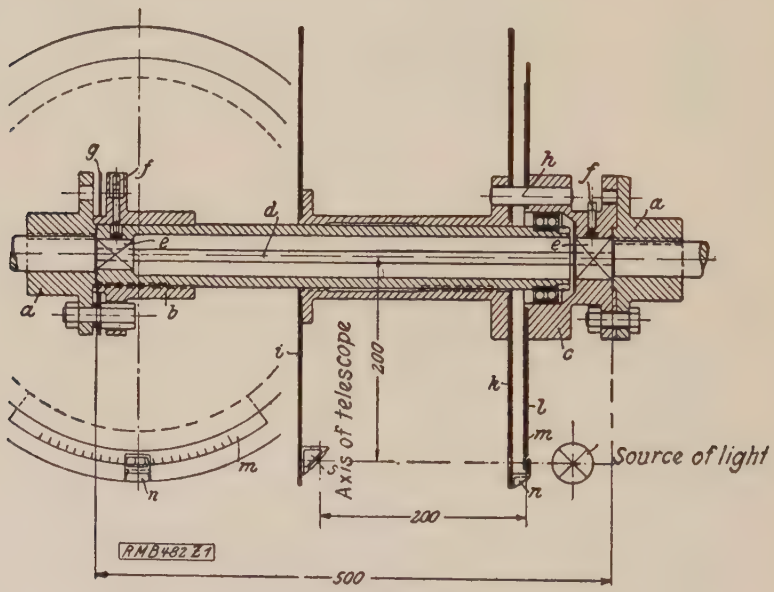


Fig. 157.

of this mirror from the scale mn equal to the distance of the mirror from the axis of the shaft. In this way a stationary image of the scale will be obtained which can be observed by telescope.*

* For the description of this instrument see the Journal "Maschinenbau" 1923–1924, p. 1028.

7. Strain Recorders.—In studying the stresses produced in engineering structures or in machine parts during vibrations, the use of special instruments, recording deformations of a very short duration, is necessary. In Fig. 158 below an instrument of this kind, the "Stress

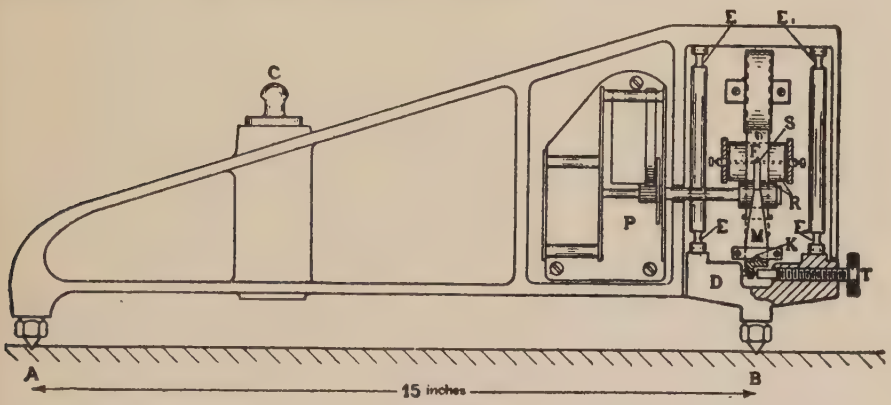


Fig. 158.

Recorder," built by the Cambridge Instrument Company, is shown.* The instrument is especially useful for the measurement of rapidly changing stresses in girders of bridges and other structures under moving or pulsating loads. To find the stress changes in a girder, the instrument is clamped to the girder under test. A clamp is placed upon the projecting part C of a spring plunger, which yields to the clamp so that the instrument is held on to the member, the extension of which is to be measured, by a pre-determined pressure. At one end of the instrument are two fixed points A, while at the other end is a single point carried on the part D, which is free to move in a direction parallel to the length of the instrument. This movement can take place because the bars E and E_1 are reduced at the points marked, the reduction in the size of the bars allowing them to bend at these points, thus forming hinges. The part D is connected to a pivoted lever M carrying the recording stylus S at its upper extremity. Any displacements of the point B due to stress changes in the structure under test are reproduced on a magnified scale by the stylus, and recorded upon a strip of transparent celluloid, which is moved past the stylus by means of a clockwork mechanism P at a rate from about 3 to 20 mm. per second. The mechanical magnification of the record in the instrument is ten. The records can be examined by

* For the description of this instrument, see "Engineering" (1924), Vol. 118, p. 287.

means of a suitable hand microscope, similar to that mentioned on page 333, or direct enlargements from the actual diagrams can be obtained by photographic methods. The record on the film can be read in this manner with an accuracy of .01 mm. Taking the distance between the points A and B equal to 15 inches, we find that the unit elongation can be measured to an accuracy of

$$\frac{.01}{10 \times 15 \times 25} = 2.66 \times 10^{-6}.$$

For steel this corresponds to a stress of 80 lbs. per square inch. The recording part of the instrument is very rigid and is suitable for vibrations of a very high frequency. For instance, vibrations of a frequency of 1400 per second in a girder have been clearly recorded but this is not necessarily the limit of the instrument. It can be easily attached to almost any part of a structure. The clockwork mechanism driving the celluloid

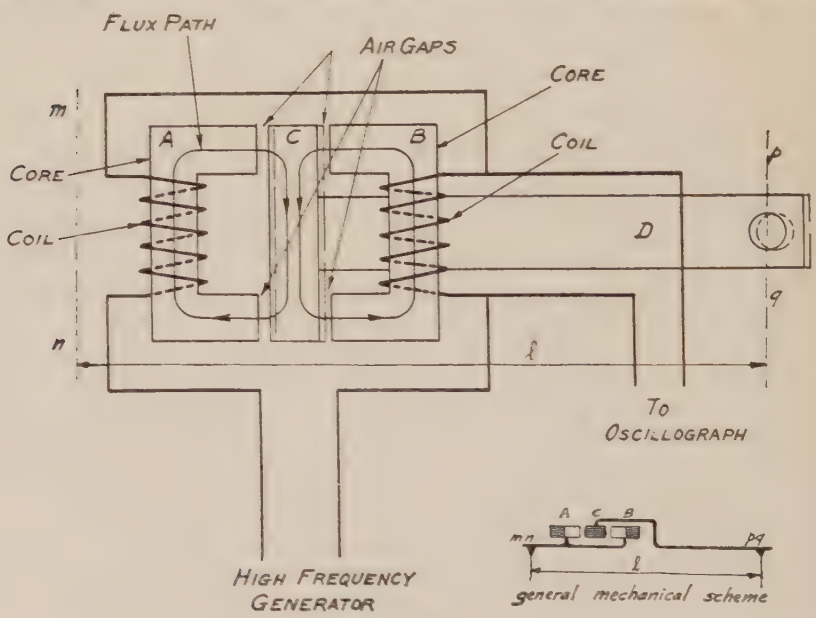


FIG. 159.

strip is started and stopped either by hand on the instrument itself or by an electrical device controlled automatically or by hand from a distance. The time-marking and position-recording mechanisms are also electrically controlled from a distance. Synchronous readings can be obtained on a

number of recorders, as they can be operated from the same time and position signals.

Figure 159 below represents the diagram of connections of a Magnetic Strain Gauge developed by Westinghouse engineers.* The instrument is held on to the member or girder, the extension of which is to be measured, by clamps such that the two laminated iron U-cores A and B forming a rigid unit are attached to the member at the cross section mn and the laminated iron yoke C through a bar D is attached at the cross section pq so that the gauge length is equal to l. Any changes in the length l due to a change in stress of the member produces relative displacements of C with respect to A and B causing a change in the air gaps. Coils are wound around the two U-shaped iron cores. Through these coils in series an a.c. current is sent of a frequency large with respect to the frequency of the stress variations to be measured. Applying a constant voltage on the two coils in series, the current taken is constant, not dependent on changes in air gap. Unequal air gaps only divide the total voltage in two unequal parts on the two coils. A record of the voltage across one coil is taken by a standard oscillograph. The ordinates of the envelope of the diagrams such as that shown in Fig. 155 are proportional to the strains in the member. This magnetic strain gauge was used † for measuring the stresses in rails, produced by a moving locomotive, and proved to be very useful. For a gauge length $l = 8$ in. an accuracy in reading corresponding to a stress of 1000 lbs. per sq. in. can be obtained.

Electric Telemeter.‡—This instrument depends upon the well known

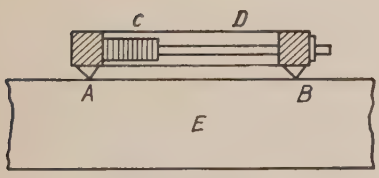


Fig. 160.

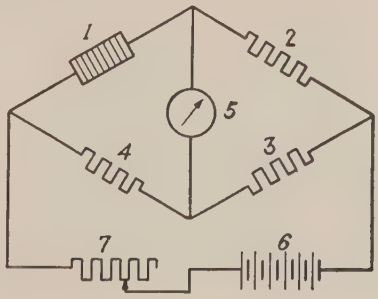


Fig. 161.

* Ritter's Extensometer.

† See writer's paper presented before the International Congress of Applied Mathematics and Mechanics. Zürich, 1926.

‡ A complete description of this instrument can be found in the technologic paper of the Bureau of Standards, No. 247, Vol. 17 (1924), p. 737, by O. S. Peters and B.

fact that if a stack of carbon discs is held under pressure a change of pressure will be accompanied by a change in electrical resistance and also a change of length of the stack. The simplest form of the instrument is shown in Figure 160 when clamped to the member E, the strain in which is to be measured. Any change in distance between the points of support A and B produces a change in the initial compression of the stack C of carbon discs, hence a change in the electrical resistance which can be recorded by an oscillograph. Fig. 161 shows in principle the electrical scheme. The instrument 1 is placed in one arm of a Wheatstone bridge, the other three arms of which are 2, 3, and 4. The bridging instrument 5, which may be a milliammeter or an oscillograph, indicates any unbalance in the bridge circuit. The resistances 2 and 3 are fixed, and 4 is so adjusted that the bridge is balanced when the carbon pile of the instrument is under its initial compression. Any change of this compression, due to strain in test member, will produce unbalance of the bridge, the extent of which will be indicated by the instrument 5, which may be calibrated to read directly the strain in the test member.

An instrument of such a simple form as described above has a defect which grows out of the fact that the resistance of the carbon pile is not a linear function of the displacement. In order to remove this defect, two carbon piles are used in actual instruments (Fig. 162). In this arrange-

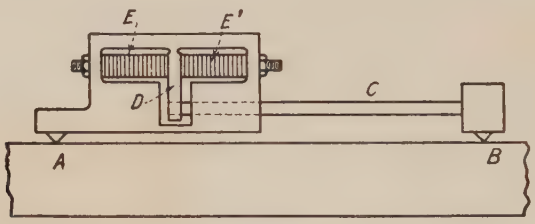


Fig. 162.

ment any change in distance between the points A and B due to strain in the test member will be transmitted by the bar C to the arm D. As a result of this an increase of compression in one of the two carbon piles E and E' and a decrease in the other will be produced. Placing these two carbon piles in the wheatstone bridge as shown in Fig. 163, the effects of changes in the resistance of the two piles will be added and the resultant effect, which now becomes very nearly proportional to the strain, will be recorded by the bridging instrument.

McCollum. See also the paper by O. S. Peters, presented before the Annual Meeting of the American Society for Testing Materials (1927).

A great range of sensitivity is possible by varying the total bridge current. Taking this current .6 amp. which is allowed for continuous operation, we obtain full deflection of the bridging instrument with .002

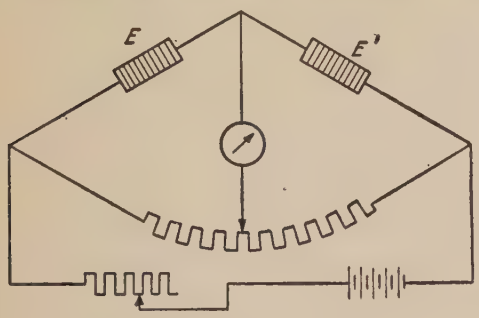


Fig. 163.

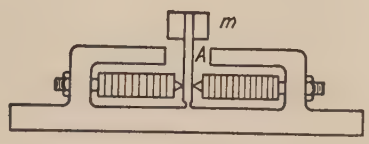


Fig. 164.

inch displacement. Hence, assuming a gauge length AB (Fig. 162) equal to 8 inches, the full deflection of the instrument for a steel member will represent a stress of about 7500 lbs. per square inch. The instrument proved to be useful in recording the rapidly varying strain in a vibrating member. Vibrations up to more than 800 cycles per second can be reproduced in true proportion.

This instrument has been also successfully used for measuring accelerations.

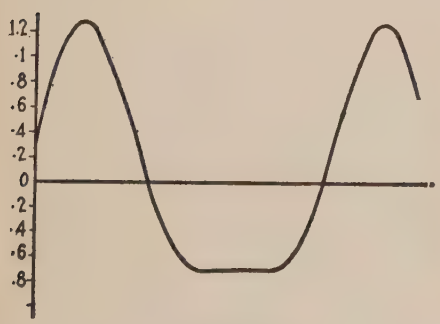


Fig. 165 *a*.

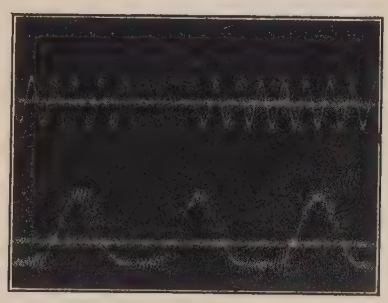


Fig. 165 *b*.

A slight modification is necessary, consisting of attaching a small mass m to the arm A (Fig. 164). The stacks act as springs, such that the natural period of vibration of the mass m is quite high (of the order of 250 per second in the experiments described below). This instrument

was mounted on an oscillating table sliding in guides and operated by a crank. The oscillation of such a table, due to the finite length of the connecting rod is not sinusoidal, but contains also higher harmonics of which the most important is the second. Fig. 165 *a* shows the acceleration diagram of this table as calculated, and Fig. 165 *b* gives the oscillograph record obtained from the carbon pile accelerometer mounted on it. The small saw teeth on this diagram have the period of natural vibration of the mass m.

AUTHOR INDEX

Akimoff, N., 43, 276
d'Alembert, 109
Amsler, E., 336

Blaess, V., 184, 190
Borden, P. A., 327
Borowicz, 171
Bourget, 305
Boussinesq, J., 285, 295
Brauchisch, E. V., 42
Bühler, A., 249

Campbell, W., 320
Clebsch, 247, 281, 284
Collins, W. G., 17, 333
Coulomb, C. A., 22
Cox, H., 281

Darnley, E. R., 234, 237
Den Hartog, J. P., 22, 191, 297
Dreyfus, L., 92
Dubois, F., 326
Duffing, G., 74, 84, 86

Eck, B., 66
Eckolt, W., 26

Föppl, A., 92
Föppl, O., 188
Fox, J. F., 152
Frahm, H., 7, 198
Freudenreich, J. von, 320, 326
Fromm, H., 22

Geiger, J., 333, 336
Goldsbrough, G. R., 159

Herz, H., 282
Holzer, H., 22, 146, 150
Hort, W., 22, 77, 271, 274, 275

Inglis, C. E., 242, 250

Karelitz, G. B., 47
Kelvin, Lord, 77
Kimball, A. L., 22
Kirchoff, G., 267, 268, 312
Kriloff, A. N., 242

Lagrange, J. L., 118
Lamb, H., 124, 125, 174, 199, 297, 316
Lamé, 301
Lehr, E., 22, 42, 154
Lemaire, P., 135
Lewis, F. M., 138, 150, 157, 220
Love, A. E. H., 199, 282, 285, 291, 293, 294, 295
Lux, F., 15

Martienssen, O., 86
Mathieu, 306
McCollum, B., 343
Meissner, E., 91, 99, 102, 107
Mises, R. von, 22
Moullin, E. B., 337
Morin, A., 22
Müller, K. E., 91, 102, 107

Navier, 285

Oehler, E., 324
Ono, A., 269, 274
Ormondroyd, J., 22, 191, 327

Pearson, 267, 281
Peters, O. S., 343
Petroff, N. P., 248
Pochhammer, L., 199, 216
Pohlhausen, E., 249
Pockels, 306
Pöschl, T., 184
Powell, J. H., 317

Ramsauer, C., 292
Rayleigh, Lord, 55, 128, 199, 227, 301, 305, 306
Reissner, H., 249

Ritter, J. G., 343
Ritz, W., 259, 264, 310, 311, 322
Roberts, J. H. T., 317
Rowell, H. S., 132, 135
Rüdenberg, R., 84

Sachs, G., 22
Sanden, H. von, 20, 48, 77, 142
Sass, F., 157
Schlick, O., 15
Schröder, P., 186
Schwerin, E., 270, 275
Sears, J. E., 292
Seefehlner, E. E., 92
Seelman, Dr., 157
Smith, D. M., 234, 237
Smith, J. H., 155
Soderberg, C. R., 44, 164
Sommerfeld, A., 36
Southwell, R. V., 317, 325
Steuding, H., 327, 337
Stodola, A., 22, 66, 159, 170, 175, 178, 184, 275, 322, 324, 325
Stokes, G. G., 248
St. Venant, 247, 281, 284, 285

Tait, W. G., 112
Thomson, J. J. Sir, 112
Timoshenko, S., 18, 80, 157, 173, 216, 227, 240, 242, 243, 245, 259, 276, 282, 343
Todhunter, 267, 281
Tolle, M., 150, 158

Vieweg, V., 337, 340
Voigt, W., 292, 310

Wagstaff, J. E. P., 292
Wallis, 248
Whipple, R. S., 337
Wiechert, A., 72, 92, 95, 108
Wrinch, Dorothy, 269
Wydler, H., 152, 154

Young, 284

Zimmermann, H., 248

SUBJECT INDEX

A

Accelerometer, 36, 345
d'Alembert's Principle, 109
Amplitude,
  definition of, 2
  frequency diagram, 29
  measurement of, 328
Automobile Vibration, 128
Axial Forces, effect on lat. vibr. of bars, 253

B

Balancing, 38
  Karelitz method of, 45
Balancing Machines
    Lawaczek-Heymann's, 40
    Akimoff's, 43
    Soderberg's, 44
Bars
  Lateral Vibrations of
    differential equation, 222
    effect of shear, etc., 226, 231
    clamped ends, 233
    free ends, 232
    hinged ends, 4, 56, 228
    cantilever, 57, 234
    one end clamped, one supported, 234
    many supports, 234
    effect of axial forces, 253
    on elastic foundation, 256
    with variable flexibility, 89
    variable section, cantilever, 264
    variable section, free ends, 270
  Longitudinal Vibrations of
    differential equation, 200
    trigonometric series solution, 200
    cantilever with loaded end, 209
    force suddenly applied, 214
    struck at the end, 286
  Forced Vibrations (supported ends)
    pulsating force, 238
    moving constant force, 241
    moving pulsating force, 245
Beats, 12, 133
Bridges, Vibration of
  moving mass, 247
  impact of unbalanced weights, 249
  irregularities of track; flats, etc., 252

C

Centrifugal Force, effect on frequency, 273
Collins Micro Indicator, 17
Conical Rod Vibration, 268
Constraint, Equations of, 109
Crank Drive, Inertia of, 158
Crank Shaft, Torsional Flexibility, 156
Critical Frequency, 9
Critical Regions, 99
Critical Speeds of Shafts
    rotating shaft, single disc, 60
    rotating shaft, several discs, 62
    analytical determination of, 159
    graphical determination of, 64, 165
    example of three bearing set, 167
    effect of gravity, 182
    gyroscopic effect, 172
    variable flexibility, 89
Critical Speed of Automobile, 135

D

Damping
  constant damping, 25
  energy absorption due to, 31
  proportional to velocity, 21
Degree of Freedom, definition, 1
Diesel Engine, Torsional vibration, 150
Discs, Turbine, 320
Dynamic Vibration Absorber, 191

E

Elastic Foundation, Vibration of bars on, 256

Energy
absorbed by damping, 31
method of calculation, 52
Equivalent Shaft Length, 146, 157

F

Flexible Bearings with rigid rotor, 182
Flywheel, gyroscopic effect of, 178
Forced Vibrations
definition, 8
with damping, 26
method of determining, 19, 51
general theory, 126
Foundation Vibration, 49
Frahm Tachometer, 15
Frame, Vibration of circular, 292, 297
Free Vibrations
definition, 1
general theory, 121
Frequency
definition, 2
measurement of, 327
equation, 123, 124
Fullarton Vibrometer, 16

G

Geared Systems, torsional vibration, 154
Generalized Coördinate, 112
Generalized Force, 114
Grooved Rotor, 66
Gyroscopic Effects, 172
on flywheels, 178

H

Harmonic Motion, definition, 3

I

Impact
lateral—on bars, 279
longitudinal—on bars, 284
Indicator, steam engines, 16, 17
Inertia of Crank Drive, 158

L

Lagrange's equations, 118
Lateral Vibration of bars, 222
Longitudinal Vibration of bars, 200
Lissajous Figures, 331

M

Membranes
general, 297
rectangular, 298
circular, 304

N

Natural Vibrations, 1
Nodal Section, 7
Normal Coördinates, 124, 127

P

Pallograph, 15
Pendulum
double, 110
spherical, 109, 119
variable length, 87
Period, definition, 2
Phase
definition, 3
with damped vibration, 27
diagram, 30
Plates
circular, 312
clamped at boundary, 314
effect of stretch of middle surface, 317
free, 317
general, 307
rectangular, 309
Pseudoharmonic Vibration
definition, 67
illustrations, 68
free, 72
forced, 81
graphical solution, 74
numerical solution, 79

Q

Quasi-harmonic Vibration
definition, 67
examples, 87

R

Rail Vibration, 256
Rayleigh Method, 55, 147

Regions of Critical Speed, 99
Resonance, definition, 9
Ring
  Complete
    radial vibration, 292
    torsional vibration, 293
    flexural vibration, 295
  Incomplete, 297
Ritz Method, 259, 302

S

Schlingertank, 198
Seismic Instruments, 14, 329
Ships, Hull Vibration of, 276
Side Rod Drive, 92
Sommerfeld's experiment, 36
Strain Recorder
    Cambridge, 341
    magnetic, 342
    telemeter, 343

T

Tachometer, Frahm, 15
Telemeter, 343
Torsiograph, 336
Torsion Meter
    Moullin, 337
    Amsler, 338
    Vieweg, 340
Torsional Vibrations
    single disc, 5
    three discs, 140
    many discs, 141
    effect of mass of shaft, 216
Turbine Blades, 270
Turbine Discs, 320

U

Unbalance, definitions
    static, 38
    dynamic, 39
Universal Recorder, 335

V

Variable Flexibility, 89
Variable Cross Section
      cantilever, 264
      free ends, 270
Vehicle Vibration, 128
Vibration Absorber, 33, 191
Vibrograph
  theory, 13, 35
  Cambridge, 332
  Geiger, 333
Vibrometer
  Fullarton, 16
  Vibration Specialty, 329
Virtual Displacement, Principle of, 109

W

Wedge, 266

Other Recent Van Nostrand Text and Reference Books for the Engineer

By CARL L. SVENSEN, *Professor of Engineering Drawing, Texas Technological College; Formerly Professor of Engineering Drawing, Ohio State University.*

Drafting for Engineers. A textbook of engineering drawing for Colleges and Technical Schools. Third Printing. 363 pp. 6¼ × 9¼. Illustrated. Cloth. $2.75.

A Handbook on Piping. A book of value to both engineers and students. 367 pp. 6 × 9. 359 Ill. 8 Folding Plates. Cloth. $4.00

Machine Drawing. A course for the correlation of drawing and engineering. 224 pp. 6 × 9. 338 Illus. 200 problems and layouts. Cloth. Fourth Printing. $2.25

Elementary Mechanism. A textbook for students of mechanical engineering, *By* A. W. STAHL, M.E., *and* A. T. WOODS, M.E., D.E. Revised and rewritten by PHILIP K. SLAYMAKER, *Professor of Machine Design, University of Nebraska.* 250 pp. 6 × 9. Illus. Cloth. $3.00

Elements of Machine Design, *By* S. J. BERARD, *Assistant Professor of Drawing and Machine Design, Brown University;* and E. O. WATERS, *Assistant Professor of Machine Design, Sheffield Scientific School, Yale University.* 316 pp. 6 × 9. Illus. Cloth. $2.50

Descriptive Geometry, *By* C. H. SCHUMANN, JR., C.E., *Assistant Professor of Drawing, Columbia University, New York. Second Printing.* 250 pp. 6 × 9. Illus. Cloth. $2.50

Controllers for Electric Motors, *By* HENRY D. JAMES, *Fellow, American Institute of Electrical Engineers; Past President, Engineers Society of Western Pennsylvania.* $5.00

Storage Batteries: Theory, Manufacture, Care and Application, *By* MORTON ARENDT, E.E., *Fellow, American Institute of Electrical Engineers; Assistant Professor of Electrical Engineering, Columbia University.* 294 pp. 156 illus. 6¼ × 9¼. Full Buckram. $4.50

Handbook of Engineering Mathematics, *By.* WALTER E. WYNNE, B.E., and W. SPRARAGEN, B.E., *Secretary, Division of Engineering and Industrial Research, National Research Council.* Second Edition—Third Printing. 285 pp. Illus. 4½ × 7. Flexible fabrikoid. $2.50

Theory of Differential Equations, *By* THORNTON C. FRY, PH.D., *Member of the Technical Staff, Bell Telephone Laboratories.* 255 pp. Cloth. 6 × 9¼. Illus. $2.50

Probability and Its Engineering Uses, *By* THORNTON C. FRY. 483 pp. Illus. 6¼ × 9¼. Full Buckram. $7.50

Introduction to Theoretical Physics, *By* LEIGH PAGE, PH.D., *Professor of Mathematical Physics, Yale University.* Second Printing. 587 pp. Illus. 6¼ × 9¼. Cloth. $6.50

Contemporary Physics, *By* KARL K. DARROW, *Member of the Technical Staff, Bell Telephone Laboratories.* Second Printing. 512 pp. Illus. 6½ × 9¼. Full Buckram. $6.00

Van Nostrand books sent post paid and on approval to residents of the United States and Canada. General Catalog sent free upon request.

15 +M✓

S0-EEA-708